25. Abduction and the Emergence of Necessary Mathematical Knowledge

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The prevailing epistemological perspective on school mathematical knowledge values the central role of induction and deduction in the development of necessary mathematical knowledge with a rather taken-for-granted view of abduction. This chapter will present empirical evidence that illustrates the relationship between abductive action and the emergence of necessary mathematical knowledge.

Recent empirical studies on abduction and mathematical knowledge construction have begun to explore ways in which abduction could be implemented in more systematic terms. In this chapter four types of inferences that students develop in mathematical activity are presented and compared followed by a presentation of key findings from current research on abduction in mathematics and science education. The chapter closes with an exploration of ways in which students can effectively enact meaningful and purposeful abductive thinking processes through activities that enable them to focus on relational or orientation understandings. Four suggestions are provided, which convey the need for meaningful, structured, and productive abduction actions. Together the suggestions target central features in abductive cognition, that is, thinking, reasoning, processing, and disposition.

25.1 An Example from the Classroom

Table 25.1 provides a short transcript of a very interesting classroom episode on counting by six that happened in a US first-grade class. The task, which was about determining the total number of faces for four separate cubes, was given to the students to help them apply and practice the arithmetical strategy of *counting on*. Anna, Betsy, and all the students together in a chorus-like manner in lines 9, 13, and 17, respectively, eagerly modeled the same process of *putting the last known number in their head and counting six more*. The episode became interesting when Ian started to employ counting by five, an arithmetical skill that the class already knew, to help him count by six in a systematic way. As conveyed in line 20, Ian initially *saw* multiples of five in the sequence (6, 12, 18, 24). In line 21, when he *added the ones* and saw that the numbers in his head matched the same numbers he saw on the teacher's board, the feeling of having *discovered* a wonderful idea caused him to exclaim *I was right!* and encouraged him to share his abduction with his classmates (lines 22–26).

Shotter [25.1] captures the following sense in which first-grade student Ian has embodied abductive thinking in relation to the number sequence (6, 12, 18, and 24): Ian was "carried away unexpectedly by an other or otherness to a place not previously familiar to him" [25.1,

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Table 25.1 Ian's counting-by-six rule

Ms. Marla [M] presented the following task below during a board math session with her first-grade class	
Number of cubes	Number of faces
1	–
2	–
3	–
4	–
How many faces do four cubes have in all?	
1	M: Let's say I have four cubes. I want to know how many faces four cubes have in
2	all. So let's count how many faces one cube would have.
3	Students [Ss]: M points to the faces one by one. One, two, three, four, five, six.
4	M: Okay so how many faces does one cube have?
5	Ss: Six!
6	M: Now I want to know how many faces two cubes would have.
7	Ss: Twelve!
8	M: Let's see. How would I figure that out?
9	Anna: Put six in your head and count six more.
10	M: Okay so?
11	Ss: 6, 7, 8, 9, 10, 11, 12.
12	M: Okay next.
13	Betsy: You put 12 in your head and count six more.
14	M: Okay everybody!
15	Ss: 12, 13, 14, 15, 16, 17, 18.
16	M: Then?
17	Ss: You put 18 in your head and count. 18, 19, 20, 21, 22, 23, 24.
18	M: So how many faces are there in all?
19	Ss: 24.
As the students began to count by six, Ian [I] decided to count by five using his right hand to indicate one set of 5.	
20	I: 5, 10, 15, 20.
He then used his right thumb and continued to count by one.	
21	I: And then you add the ones. 21, 22, 23, 24. I was right!
Ian eagerly raised his hand and shared his strategy with Ms. Marla and his classmates.	
22	I: Ms. M, I was thinking that in my head. ... Ms. M I know another idea ...
23	because you have all those sixes and you count by fives and there's only ones
24	left.
25	M: So you went 5, 10, 15, 20. [Ian nods].
26	I: 21, 22, 23, 24.
27	M: Excellent!

p. 225]. He was pleasantly surprised about how easy it was to count "all the sixes" by "counting by fives and adding the ones left", which generated in him an intense feeling of discovering something new through a guess that made sense and that he was able to verify to be correct. The following passage below from *El Khachab* [25.2] provides another, and yet deeper, way of thinking about Ian's experience. El Khachab foregrounds the significance of having a *purpose* as a way of motivating the emergence of new ideas, which is one way of explaining how learners sometimes find themselves being carried away during the process of discovery. The second sentence in the passage articulates in very clear terms the primary purpose of abduction and its central and unique role in the establishment of new knowledge [25.2, p. 172]:

> "Before asking where new ideas come from, we need to ask what new ideas are for, and knowing what they are for, we can attune their newness to their purpose. And their purpose is, in the case of abduction, to provide true explanations following experimental verification."

Ian saw purpose in counting by *five plus one* that encouraged him to further pursue his *new* idea. After verifying that his strategy actually worked on the available cases, he then articulated an explanation that matched what he was *thinking in his head*. The nature of what counts as a *true explanation* in abduction is explored in some detail in the succeeding sections. For now, it makes sense to think of abductive explanations as modeling instances of "relational or orientational

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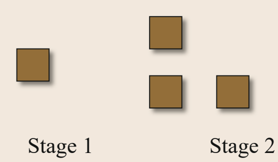
way of knowing", which is a type of "embodied coping" that attends to [25.2, p. 172]

> "the possible relations – what we might call the *relational dimensions* – that exist as a dynamical outcome of the interacting of objectively observable phenomena which are not in themselves objectively observable."

Ian's abductive thinking about counting by six is worth noting early in this chapter in light of recent findings on children's algebraic thinking that show how many of them tend to use their knowledge of the multiplication table to help them generate and establish mathematical relationships and support their ability to construct explicit or function-based formulas involving linear patterns [25.3].

US eighth-grade student Dung's figural processing of the two pattern generalization tasks shown in Figs. 25.1 and 25.2 illustrates another characterization of abductive thinking that "carries over a deeper similarity to a number of seemingly rather different situations" [25.1, p. 225]. Dung's processing illustrates a kind of *double description* (i. e., in *Bateson*'s [25.5, p. 31] sense of "cases in which two or more information sources come together to give information of a sort different from what was in either source separately") that is a necessary condition when students are engaged in mathematical thinking and learning. When Dung was presented with the ambiguous Fig. 25.1 task consisting of two beginning stages in a growing pattern, he constructed a growing sequence of L-shaped figures (Fig. 25.3). When he was asked to generate explicit rules for his pattern, he suggested $s = n + n - 1$ and $s = 2n - 1$. When he was asked to justify them, Dung saw the pattern stages in terms of groups of squares. In the case of his first rule, each stage in his growing pattern consisted of the union of two variable units having cardinalities n and $(n - 1)$ corresponding to the column and row of squares, respectively (see Fig. 25.3 stage 3 for an illustration). In the case of his second rule, two composite sides of squares that had the same number of squares on each side overlapped along the corner square (see Fig. 25.3 stage 5 for an example).

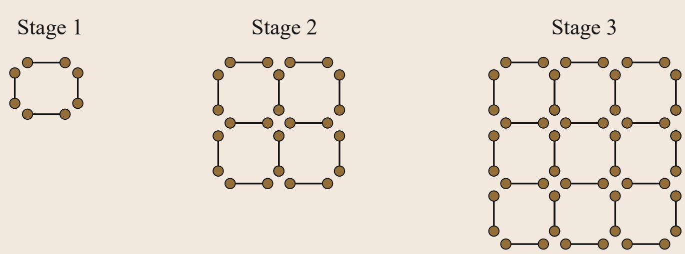
Below are the first two stages in a growing pattern of squares



1. Continue the pattern until stage 5.
2. Find a direct formula in two different ways. Justify each formula.
3. If none of your formulas above involve taking into account overlaps, find a direct formula that takes into account overlaps. Justify your formula.
4. How do you know for sure that your pattern will continue that way and not some other way?
5. Find a different way of continuing the pattern and obtain a direct formula for this pattern.

Fig. 25.1 Ambiguous patterning task in compressed form (after [25.4])

Consider the following array of sticks below



A. Find a direct formula for the total number of sticks at any stage in the pattern. Justify your formula.
B. Find a direct formula for the total number of points at any stage in the pattern. Justify your formula.

Fig. 25.2 Square array pattern (after [25.4])

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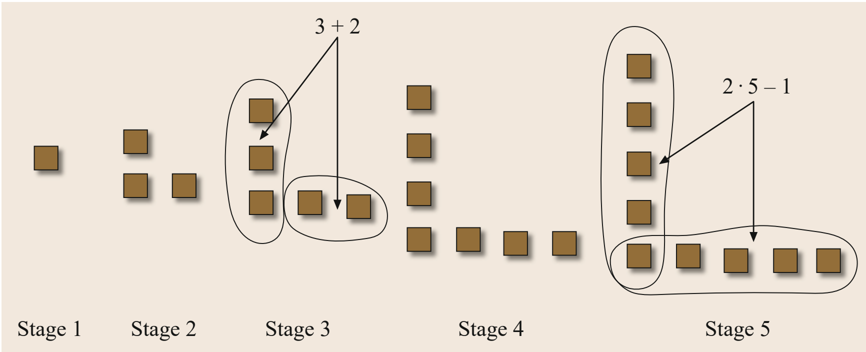

Fig. 25.3 Dung's growing L shaped pattern (after [25.4])

For Dung, seeing pattern stages in terms of groups enabled him to justify his explicit rules, which became his abductive resource for constructing and justifying an explicit rule for the square array pattern shown in Fig. 25.2. Dung initially saw each pattern stage into parts of separate rows of squares and separate smaller squares per row (Fig. 25.4). Using stage 4, he parsed the whole figure into four disjoint rows and counted the number of sticks per row. In counting the number of sticks per row, he saw four disjoint squares for a total of $4 \times 4 = 16$ sticks and then subtracted the three overlapping vertical sticks. He then counted the total number of horizontal and vertical sticks counting repetitions and obtained $(4 \times 4) \times 4 = 52$. In his written work, he immediately resorted to the use of a variable n to convey that he was thinking in general terms, which explains the expression $(4n - (n-1)) \times n$. Since he also saw that the four disjoint rows had overlapping sides (i. e., the interior horizontal sticks), he then took away three $(= 4-1)$ groups of such four horizontal sticks from 52. That concrete step allowed him to complete his explicit rule for the pattern, that is, $s = (4n - (n-1))n - (n-1)n$, which he then simplified to $s = 2n^2 + 2n$. Dung's multiplicative thinking ability became his abductive – that is, double descriptive – abstracting resource that enabled him to infer deeper similarity among, and thus generalize to, different kinds of patterns.

In this chapter, we explore the relationship between abductive action and the emergence of necessary mathematical knowledge. The prevailing epistemological perspective on mathematical knowledge values the central role of induction and deduction in the development of necessary mathematical knowledge with a rather taken-for-granted view of abduction that in the past has been characterized as the creative, wild, and messy space of theory generation or construction. However, recent empirical studies on abduction and mathematical knowledge construction have begun to explore ways in which abduction could be implemented in more systematic terms beyond a way of reasoning by detectives from observations to explanations [25.6, p. 24] and merely "studying facts and devising a theory to ex-

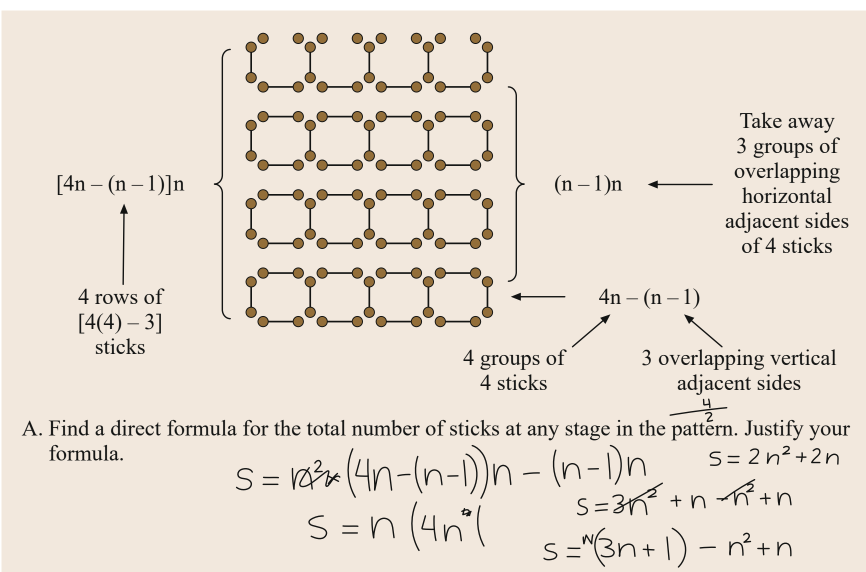

Fig. 25.4 Dung's construction and justification of his formula for the Fig. 25.2 pattern (after [25.4])

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plain them" because "its only justification is that if we are ever to understand things at all, it just be in that way" [25.7, p. 40]. For instance, *Mason* et al. [25.8] associate abductive processing with the construction of structural generalizations, while *Pedemonte* [25.9] situates abduction within a cognitive unity thesis that sees it as being prior and necessary to induction and ultimately deduction. Recent investigations in science and science education that pursue an abductive framework also underscore the central role of abduction in inference systems that model everyday phenomena. For instance, Addis and Gooding propose the iterative cycle of "abduction (generation) → deduction (prediction) → induction (validation) → abduction" in modeling the "scientific process of interpreting new or surprising findings by generating a hypothesis whose consequences are then evaluated empirically" [25.10, p. 38]. Another instance involves *Magnani*'s [25.11] formulation of actual computational models in which case abduction is seen as central to the development of creative reasoning in scientific discoveries and can thus be used to generate rational models.

In Sect. 25.2, we provide a characterization of the four types of inferences that students develop in mathematical activity. In Sect. 25.3 we note two key findings from current research on abduction in mathematics and science education, which should provide the necessary context for understanding the ideas we pursue in the succeeding section. In Sect. 25.4 we explore ways in which students can effectively enact meaningful and purposeful abductive thinking processes and other [25.1, p. 224]

> "kinds of *preparing activities* in mathematical learning contexts that will enable learners to become self-consciously engaged in, can get them *ready* to notice, immediately and spontaneously, the kinds of events relevant to their acquiring such relational or orientation understandings – where, by *being ready to do something* means what we often talk of as being in possessions of a *habit*, an *instinct*, an *inclination*, etc."

Central to such processes and activities involves orchestrating effective tasks and other learning contexts that will engage all students in abductive thinking, which will go a long way in supporting growth in necessary mathematical knowledge and excellence in reasoning that is strategic and has "logical virtue (i. e., avoiding logical fallacies and learning what is and what is not admissible and valid)" [25.12, p. 269].

25.2 Inference Types

Table 25.2 lists the characteristics of four types of inferences that students develop in mathematical activity. *Abduction* involves generating a hypothesis or narrowing down a range of hypotheses that is then verified via *induction*. Abduction is the source of original ideas and is initially influenced by prior knowledge and experiences, unlike induction that basically tests an abductive claim on specific instances. The hope, of course, is that possible errors get corrected through the inductive route, which results in the construction of a generalization that draws on the available instances. Like induction, which performs the role of verifying an abductive claim, *deduction* produces results from general rules or laws and thus does not produce any original ideas. Unlike abduction, which is sensitive to empirical data, deduction relies on unambiguous premises in order to ascertain the necessity of a single valid conclusion. An *unambiguous* well-defined set or model assumes the existence of "a finite set of rules and without reference to context" that clearly defines membership or relationships among the elements in the set [25.10, p. 38].

Another useful way to think about abduction and deduction involves truth tables. Deductions depend on truth tables for validity, which also means to say that the objects and rules of deduction all have to be well established and well defined. Abductions do not depend on truth tables and their validity is established via induction [25.10, p. 37–38]. *Deductive closure* conveys deductively derived arguments and instances and is a necessary condition for algebraic thinking in both symbolic and nonsymbolic contexts [25.13].

Consider, for example, the following four statements below that have been extracted from eighth-grade student Cherrie's generalization of the Fig. 25.3 pattern:

Law (L): I think the rule is $x = 2(n+1)n$.
Case (C): In stage 2, there's two groups of three twice. There's two four groups of three in stage 3. There's two five groups of four in stage 4.
Result (R): Stages 1 through 4 follow the rule x equals two times $(n+1)$ times n.
All future outcomes (O): Stage 5 has $26(5) = 60$ sticks, stage 10 has $211(10) = 220$ sticks, and stage 2035 has 8 286 520 sticks in all.

Deduction assumes a general law and an observed case (or cases) and infers a necessary valid result, which also means that it does not have to depend on real or empirical knowledge for verification [25.14]. Cases

Table 25.2 Types of inferences in mathematical activity and their characteristics

Inferential type	Inferential form	Intent	Inferential attitude	Sources	Desired construction	Nature of context, verification, and justification
Abduction	From result and law to case	Depth (intentional)	Entertains a plausible inference toward a rule Generates and selects an explanatory theory – that *something maybe* (conjectural)	Unpredictable (surprising facts; flashes; intelligent guesses; spontaneous conjectures)	Un/ Structured	Context-bound; Structured via induction
Induction	From result and more cases to law	Breadth (extensional)	Tests an abduced inference; measures the value and degree of concordance of an explanatory theory to cases – that *something actually is operative* (approximate)	Predictable (examples)	Structural based on abduction	Context-derived; empirical (e.g., enumeration, analogy, and experiments)
Deduction	From rule and case to result	Logical proof	Predicts in a methodical way a valid result – that *something must be* (certain)	Predictable (premises)	Structural (canonical form)	Decontextualized; Steps in a proof
Deductive closure	From an established deduction to future outcomes	Breadth (apply)	Assumes that all future outcomes will behave in the same manner as a result of a valid deductive hypothesis	Predictable (premises are valid deductions)	Structural based on an established deduction	Decontextualized; Mathematical induction (e.g., demonstration of a valid deductive claim)

are occurrences or instantiations of the stipulated law. When the first three statements above are switched in two different ways, we obtain the canonical structures for abduction and induction, which are ampliative because the conclusions "amplify or go beyond the information incorporated in the premises" [25.11, p. 511] and invalid (i. e., not necessary) from a deductive point of view. In a deductive closure, an established deduction becomes the cause or hypothesis that is then applied to future outcomes, which are effects. Figure 25.5 visually captures the fundamental differences among the four inferential types.

From a logicopsychological perspective, students need to learn to anticipate inferences that are sensible and valid in any mathematical activity. *Peirce* [25.15, p. 449], of course, reminds us that context matters despite our naturally drawn disposition toward "perpetually making deductions". As an aside, kindergarten students (ages five to six years) in the absence of formal learning experiences appear to consider deductive inferences as being more certain than inductive ones and other guesses [25.16].

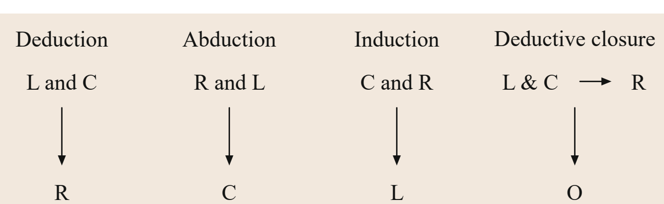

Fig. 25.5 Differences among the four inferential types

Students also need to understand the limitations of each inferential process. For *Polya* [25.17], deduction exemplifies *demonstrative reasoning*, which is the basis of the "security of our mathematical knowledge" [25.17, p. v] since it is "safe, beyond controversy, and final". Abduction and induction exemplify *plausible reasoning*, which "supports our conjectures" and could be "hazardous, controversial, and provisional" [25.17]. Despite such constraints, however, Peirce and Polya seem to share the view that abduction, induction, and deduction are epistemologically necessary. According to *Polya* [25.17], while "anything new that we learn about the world involves plausible reasoning", demonstrative reasoning uses "rigid standards that are codified and clarified by logic" [25.17, p. v]. Polya's perspectives are narrowly confined to how we come to understand and explain the nature of mathematical objects, unlike Peirce who formulates his view by drawing on his understanding of the nature of scientific practice. "All ideas of science come to it by way of abduction", *Peirce* writes, which is the fundamental source of the emergence of ideas and "consists in studying facts and devising a theory to explain them" [25.7, p. 90].

In the next three subsections below, we discuss additional characteristics of each inferential type.

25.2.1 Abduction

Abduction, the source of original ideas, discoveries, and explanatory theories, emerges and evolves in a continuum of thought processes, uncontrolled and instinctual (e.g., as initial impressions based on perceptions and informed guesses) in the early phase and structured and inferential (e.g., quasideductive) in a much later phase [25.2]. Through abduction, "descriptions of data, patterns, or phenomena" are inferred leading to plausible explanations, hypotheses, or theories "that deserve to be seriously entertained and further investigated" ([25.18, p. 1021], [25.11, p. 511]). Perceptual-like clues provide one possible source of abductive ideas [25.19]. The steps below outline a percept-based "formula that is similar to abduction" [25.19, p. 305].

> "A well-recognized kind of object, M, has for its ordinary predicates P[1], P[2], P[3], etc., indistinctly recognized. The suggesting object, S, has these same predicates, P[1], P[2], P[3], etc. Hence, S is of the kind M."

Iconic-based inferences also provide another possible source of abduction [25.19]. Icons, unlike percepts, are pure possible forms of the objects they represent or resemble. Iconic-based abductions employ the following abductive process [25.19, p. 306]:

$$\left.\begin{matrix}\text{P1}\\ \text{H1}\end{matrix} \rightarrow \right\} \text{An iconic relationship between P1 and P2}$$

P1 and P2 are similar (iconically)

$\therefore$ Maybe H1 (or something that is similar to H1) .

Abduction also involves "the problem of logical goodness, i. e., how ideas fulfill their logical purpose in the world" [25.2, pp. 159, 162]. *El Khachab* [25.2] uses the example of global warming to show how different stakeholders tend to model different kinds of goodness based on their purpose. Following Peirce, he notes that "the purpose of abduction is to provide hypotheses which, when subjected to experimental verification, will provide true explanations" [25.2, p. 162]. True explanations refer to "sustainable belief-habits, that is, as recurring settlements of belief about the world which rely on experientially or experimentally verifiable statements" [25.2, p. 163].

We note the following four important points below about abduction.

First, *Tschaepe* [25.20] underscores the significance of guessing in abduction, that is [25.20, p. 117],

> "guessing is the initial deliberate originary activity of creating, selecting, or dismissing potential solutions to a problem as a response to the surprising experience of that problem."

Having a guess enables learners to transition from the first to the second premise in Peirce's general syllogism for abduction (i. e., the surprising fact, C, is observed; but if A were true, C would be a matter of course; hence, there is reason to suspect that A is true). Following *Kruijff* [25.21], Tschaepe notes that guessing and perceptual judgment (i. e., observing a surprising fact C) are "*the two essential aspects that characterize* the generation of ideas" [25.20, p. 117], where the event of surprise emerges from every individual knower's experiences, which is perceptual in nature. Guessing, then [25.20, p. 117],

> "follows perceptual judgment, signifying a transition between uncontrolled thought and controlled reasoning. [. . .] We guess in an attempt to address the surprising phenomenon that has led to doubt; it is our inchoate attempt to provide an explanation."

Second, *Thagard* [25.22] makes sense in saying that an abductive process involves developing and entertaining inferences toward a law that will be tested via induction, which will then produce inferences about a case. For *Eco* [25.23], however [25.23, p. 203],

> "the real problem is not whether to find first the Case or the Law, but rather how to figure out both the Law and the Case *at the same time*, since they are inversely related, tied together by a sort of chiasmus."

Third, while the original meaning of abduction based on Peirce's work refers to inferences that yield plausible or explanatory hypotheses, *Josephson* and *Josephson*'s [25.24] additional condition of *inferences that yield the best explanation* revises the structure of the original meaning of abduction in the following manner:

Case: D is a collection of data (facts, observations, givens).
Law: H explains D (would, if true, explain D).
Strong Claim: No other hypothesis can explain D as well as H does.
Result: H is probably true.

Paavola [25.25] notes that while the original and revised versions of abductions share the concern toward generating explanations, they are different in several ways. The original version addresses issues related to the *processes of discovery* and the construction of plausible hypotheses, while the revised version models a nondeductive form of reasoning (except induction) that eventually establishes the true explanation. Across

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the differences, it is instructive to keep in mind both *Adler*'s "simple, conservative, unifying, and yields the most understanding" conditions for constructing strong abductions [25.26, p. 19] and *El Khachab*'s logical goodness conditions that characterize good abductions. That is, they [25.2, p. 164]

> "(1) need to be clear, i. e., they need to have distinguishable practical effects; (2) they need to explain available facts; and (3) they need to be liable to future experimental verification."

Fourth, it is important to emphasize that abductions provide explanations or justifications that do not prove. Instead, they provide explanations or justifications that primarily assign causal responsibility in *Josephson*'s [25.27, p. 7] sense below.

> "Explanations give causes. Explaining something, whether that something is particular or general, gives something else upon which the first thing depends for its existence, or for being the way that it is. [...] It is common in science for an empirical generalization, an observed generality, to be explained by reference to underlying structure and mechanisms."

25.2.2 Induction

Unlike abduction, induction tests a preliminary or an ongoing abduction in order to support a most reasonable law and thus develop a generalization that would both link and unite both the known and projected cases together in a meaningful way. By testing an abductive claim over several cases, induction determines whether the claim is right or wrong. So defined, a correct induction does not produce a new concept that explains (i. e., an *explanatory theory*), which is the primary purpose of abductive processing. Instead, it seeks to show that once the premises hold (i. e., the case/s and the result/s), then the relevant conclusions (i. e., the law) must be true by enumeration (number of observed cases), analogy (i. e., structural or relational similarity of features among cases), or scientific analysis (through actual or mental experiments) [25.28] and thus reflect causal relationships that are expressed in the form of (categorical inductive or universally quantified) generalizations [25.11]. In the case of enumeration, in particular, the goal is not to establish an exhaustive count leading to a precise numerical value, but it is about "producing a certain psychological impression [...] brought about through the laws of association, and creating an expectation of a continuous repetition of the experience" [25.28, p. 184]. In all three contexts of inductive justification, inductive inferences do not necessarily yield true generalizations. However, "in the long run they approximate to the truth" [25.29, p. 207].

Four important points are worth noting about the relationship between abduction and induction, as follows:

First, *El Khachab* points out how both abduction and induction appear to be "unclear" about their "practical effects which are essentially similar" [25.2, p. 166]. However, they are different in terms of "degree", that is [25.2, p. 166],

> "an induction is an inference to a rule; an abduction is an inference to a rule *about* an occurrence, or in Peirce's own words, *an induction from qualities* [...] Induction is a method of experimental verification leading to the establishment of truth in its long-term application."

Second, abduction is not a requirement for induction. That is, there can be an abduction without induction (i. e., abductive generalizations). Some geometry theorems, for example, do not need inductive verification. In some cases, abduction is framed as conjectures that are used to further explain the development of schemes ([25.30] in the case of fractions). However, it is useful to note the insights of *Pedemonte* [25.9] and *Prusak* et al. [25.31] about the necessity of a structural continuity between an abduction argument process and its corresponding justification in the form of a logical proof. That is, a productive abductive process in whatever modal form (visual, verbal) should simultaneously convey the steps in a deductive proof.

Even in the most naïve and complex cases of inductions (e.g., number patterns with no meaningful context other than the appearance of behaving like objects in some sequence), learners initially tend to produce an abductive claim as a practical embodied coping strategy, that is, as a way of imposing some order or structure that may or may not prove to make sense in the long haul. Euler's numerical-driven generalization of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a good example. He initially established an analogical relationship between two different types of equations (i. e., a polynomial P of degree n having n distinct nonzero roots and a trigonometric equation that can be transformed algebraically into something like P but with an infinite number of terms). Euler's abductive claim had him hypothesizing an anticipated solution drawn from similarities between the forms of the two equations. Upon inductively verifying that the initial four terms of the two equations were indeed the same, Euler concluded that [25.17, pp. 17–22]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

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Third, another consequence of the preceding discussion involves the so-called *inductive leap*, which involves establishing a generalization from concrete instances to a conclusion that seems to contain more than the instances themselves. On the basis of the characterizations we have assigned to abduction and induction, such a leap is no longer an issue since the leap itself is settled by abduction. Hence, criticisms that in effect cite "hazardous inductive leap" as an argument in relation to erroneous patterning questions such as the one shown in Fig. 25.6 is more appropriately and fundamentally a problem of abduction.

Fourth, neither abduction nor induction can settle the issue of *reasonable of context*. For example, the patterning situation in Fig. 25.7 can have a stipulated abduction and an inductively verified set of outcomes based on an interpreted explicit formula. However, as *Parker* and *Baldridge* [25.32] have noted, "there is no reason why the rainfall will continue to be given by that expression, or *any* expression", which implies that the "question cannot be answered" [25.32, p. 90].

25.2.3 Deduction and Deductive Closure

While abduction and induction provide support in constructing or producing a theory, both deduction and deductive closure aim to exhibit necessity. *Pace Smith* [25.33]: "(R)epeated co-instantiation via induction is not the same as inferential necessity" [25.33, p. 5]. A valid deduction demonstrates a logical implication, that is, it shows how a law and a case as premises or hypotheses together imply a necessary result, conclusion, or consequence. It is a "self-contained process" because the validation process relies on "the existence of well-defined sets" and preserves an already established law, thus, "freeing us from the vagaries and changeability of an external world" [25.10, p. 37].

A certain pattern begins with 1, 2, 4. If the pattern continues, what is the next number?

A. 1
B. 2
C. 7
D. 8

Fig. 25.6 An example of an erroneous generalization problem

Deductive closure emerges in students' mathematical thinking and reasoning in at least two ways depending on grade-level expectations, as follows. Among elementary and middle school students, once they (implicitly) form a deduction, they tend to provide an empirical (numerical or visual) structural argument then a formal deductive proof as a form of explanation or justification. For example, Cherrie's algebraic generalization relative to the pattern in Fig. 25.3 could be expressed in deductive form. When she began to correctly apply her result to any stage in her pattern beyond the known ones, her reasoning entered the deductive closure phase.

Among high school students and older adults, once they formulate a deduction, they tend to provide any of the following types of justification that overlap in some situations: an empirical structural argument; a logical deductive proof; or a mathematical induction proof. Figure 25.8 illustrates how a group of 34 US Algebra 1 middle school students (mean age of 13 years) empirically justified the fact that $-a\times -b = -(a\times -b)$by demonstrating a numerical argument following a statement-to-reason template [25.34, pp. 126–130]. Note that when the numbers in the empirical argument shown in Fig. 25.8 are replaced with variables, the argument transforms into a logical deductive proof in which case the steps follow a logical "recycling process" (Duval, quoted in *Pedemonte* [25.9, p. 24]), that is, the conclusion of a foregoing step becomes the premise of a succeeding step from beginning to end. Deductive closure for these students occurred when they began to obtain products of integers (and, much later, rational numbers) involving negative factors without providing a justification.

Figure 25.9 shows a mathematical inductive proof of a classic theorem involving the sum of the interior angles in an n-sided convex polygon that has been drawn from *Pedemonte*'s [25.9] work with 102 Grade 13 students (ages 16–17 years) in France and in Italy. The "multimodal argumentative process of proof" [25.31, 35] evolved as a result of a structural continuity between a combined abductive-inductive action that was performed on a dynamic geometry tool, which focused on a perceived relationship between the process of constructing nonoverlapping triangles in a polygon and the effects on the resulting interior angle sums, and the accompanying steps that reflected the structure of a mathematical induction proof.

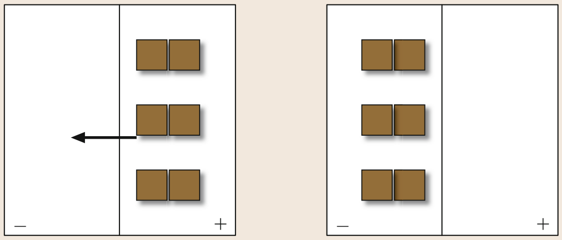
It started to rain. Every hour Sarah checked her rain gauge. She recorded the total rainfall in a table. How much rain would have fallen after h hours?

Hours	Rainfall
1	0.5 in
2	1 in
3	1.5 in

Fig. 25.7 An Example of a patterning task with an erroneous context (after [25.32])

Part E | 25.2

Based on the figure below, Let us illustrate why –3 × 2 = –(3 × 2) using properties of integers. –3 × 2 = –(3 × 2) means pull 3 groups of 2 cubes on the positive region to the negative region



–3 × 2	=	*(–3 × 2) + 0*	*Additive identity property*
	=	*(–3 × 2) + [(3 × 2) + –(3 × 2)]*	*Additive inverse property*
	=	*[(–3 × 2) + (3 × 2)] + –(3 × 2)*	*Associative property*
	=	*[(–3 + 3) × 2] + –(3 × 2)*	*Distributive property*
	=	*0 + –(3 × 2)*	*Additive inverse property*
	=	*–(3 × 2)*	*Additive identity property*

Fig. 25.8 An empirical structural argument for $-a \times -b = -(a \times -b)$ (after [25.34])

66. M: If n is equal to 3, f(n) is equal to 180 × 1... If n is equal to 4, f(n) is equal to 360, which is equal to 180 × 2
67. L: N equal to 5, f(n) is equal to 540, which is equal to 180 × 3...
68. M: So f(n) is equal to 180 × (n–2) ...
69. L: OK, now we have to understand why ...

70. M: OK... wait!
71. L: F(4) is equal to 180 + f(3) because there is one triangle more... so 180 + 180...
72. M: OK, then f(5) is... is f(4) + 180... that means that f(n) is equal to f(n – 1) + 180
73. L: You always add 180 to the previous one
74. M: OK we can write f(n + 1) as f(n) + 180...

Base F(3) = 180°

F(n + 1) = 180°(n – 1)

F(n + 1) = F(n) + 180°
It is necessary to add 180° to F(n) because if we add a side to the polygon, we add a triangle too. The sum of the triangles angles is 180°.
So:

F(n + 1) = 180°(n – 2) + 180°
F(n + 1) = 180°(n – 2 + 1)
F(n + 1) = 180°(n – 1)

Fig. 25.9 A mathematical inductive proof for the sum of the interior angles in an n-sided convex polygon (after [25.8, p. 37–38])

The work shown in Fig. 25.10 was also drawn from the same sample of students that participated in *Pedemonte*'s [25.9] study. Unlike Fig. 25.9, the analysis that the students exhibited in Fig. 25.10 shows a structural discontinuity between a combined abductive-inductive action, which primarily focused on the results or outcomes in a table of values, and steps that might have produced either a valid empirical justification or a logical mathematical induction proof. Deductive closure for these students occurred when they began to obtain the interior angle sum measures of any convex polygon beyond the typical ones.

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Alice constructs the following table:

Sides	Sum (Angles)	
3	180°	
4	360°	180° × 2
5	540°	180° × 3
6	720°	180° × 4

29. A: So the rule is probably 180 × (n–2) for an n-sided polygon
30. L: Yes... n is the number of sides

Base for n = 3
180°(3–2) = 180°

Step
Hp: 180°(n–2)
Ts: 180°(n–1)

S(n) = 180° (n–2) = 180n – 360
S(n+1) = 180° (n+1) – 360 = 180n + 180 – 360 = n + 1 – 2 = n – 1 Th
We have proved the thesis by a mathematical induction

Fig. 25.10 Example of an erroneous mathematical inductive argument for the sum of the interior angles in an *n*-sided convex polygon (after [25.8, p. 36])

25.3 Abduction in Math and Science Education

A nonexhaustive survey of recent published studies dealing with abduction in mathematical and scientific thinking and learning yields two interesting findings, as follows.

25.3.1 Different Kinds of Abduction

Drawing on *Eco*'s [25.23] work, *Pedemonte* and *Reid* [25.36] provided instances in which traditional 15–17-year-old Grades 12 and 13 students in France and Italy modeled overcoded, undercoded, and creative abductions in the context of proving statements in mathematics. For Pedemonte and Reid, abduction comes before deduction. Some students in their study generated overcoded abductions, which involve using a single rule to generate a case, while others produced undercoded abductions, which involve choosing from among several different rules to establish a case. Overcoded and undercoded abductions for *Magnani* [25.11] exemplify instances of selective abductions because the basic task involves selecting one rule that would make sense, which, hopefully, would also yield the best explanation. Medical diagnosis, for instance, employs selective abductions [25.11]. In cases when no such rules exist, students who develop new rules of their own yield what *Eco* [25.23] refers to as creative abductions, which also account for "the growth of scientific knowledge" [25.11, p. 511]. Pedemonte and Reid have noted that students are usually able to construct a deductive proof in cases involving overcoded abductions due to the limited number of possible sets of rules to choose from. Furthermore, they tend to experience considerable difficulties in cases that involve undercoded and creative abductions since they have to deal with "irrelevant information in the argumentation process, thus confusing, and creating disorder" in their processing [25.36, p. 302]. An additional dilemma that students have with creative abductions is the need to justify them prior to using them as rules in a proof process. "Consequently", *Pedemonte* and *Reid* write [25.36, p. 302],

> "it seems that there is not a simple link between the use of abduction in argumentation and constructing a deductive proof. Both the claim that abduction is an obstacle to proof and the claim that abduction is a support, if considered in a general sense, are oversimplifications. Some kinds of abductions, in some context may make the elements required for the deductions used in a proof more accessible. Some are probably less dangerous to use and can make the construction of a proof easier to get to because they could make easier to find and to select the theorem and the theory necessary to produce a proof. However, other kinds of abductions present genuine obstacles to constructing the proof. This suggests that teaching approaches that involve students conjecturing in a problem solving process prior to proving have potential, but great care must be taken that the abductions expected of the students do not become obstacles to their later proving."

Aside from selective and creative abductions, *Magnani* [25.11] pointed out the significance of theoretical and manipulative abductions in other aspects of everyday and scientific work that involve creative processing. Theoretical abductions involve the use of logical, verbal or symbolic, and model-based (e.g., diagrams and pic-

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tures) processing in reasoning. While valuable, they are unable to account for other possible types of explanations (e.g., statistical reasoning, which is probabilistic; sufficient explanations; high-level kinds and types of creative and model-based abductions; etc.). Manipulative abductions emerge in cases that involve "thinking and discovering through doing", where actions are pivotal in enabling learners to model and develop insights simultaneously leading to the construction of creative or selective abductions. They operate beyond the usual purpose of experiments and create "extra-theoretical behaviors" that [25.11, p. 517]

> "create communicable accounts of new experiences in order to integrate them into previously existing systems of experimental and linguistic (theoretical) practices. The existence of this kind of extra-theoretical cognitive behavior is also testified by the many everyday situations in which humans are perfectly able to perform very efficacious (and habitual) tasks without the immediate possibility of realizing their conceptual explanation."

Typical accounts of conceptual change processes in science tend to highlight theoretical abductions, however [25.11, p. 519],

> "a large part of these processes are instead to due practical and *external* manipulations of some kind, prerequisite to the subsequent work of theoretical arrangement and knowledge creation."

Manipulative abductions may also emerge in learning situations that provide "conceptual and theoretical details to already automatized manipulative executions" in which case either teacher or learner [25.11, p. 519]

> "does not discover anything new from the point of view of the objective knowledge about the involved skill, however, we can say that his conceptual awareness is new from the local perspective of his individuality."

For example, *Rivera* [25.37] provides a narrative account of US third-grade Mark's evolving understanding of the long division algorithm involving multidigit whole numbers by a single-digit whole number. Mark's initial visual representation processing of (sharing-partitive) division (Fig. 25.11) employed the use of place value-driven squares, sticks, and circles. In the case of the division task $126 \div 6$, when he could not divide a single (hundreds) box into six (equal) groups, he recorded it as a 0. He then ungrouped the box into ten sticks, regrouped the sticks together, divided the

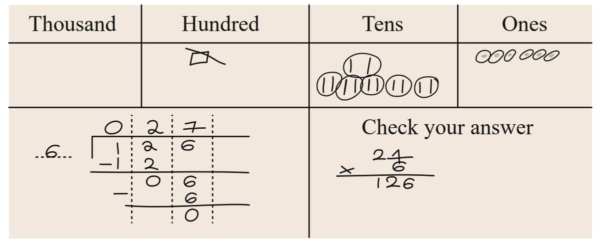

Fig. 25.11 Mark's initial visual processing of $126 \div 6$

sticks into six groups, recorded accordingly, and so on until he completed the division process for all subcollections. His numerical recording in Fig. 25.11 also captured every step in his sequence of visual actions. Results of consistent visual processing enabled him to shift his attention away from the visual form and toward the rule for division, which was accompanied by two remarkable changes in his numerical processing. In Fig. 25.12, he performed division on each digit in the dividend from left to right with the superscripts indicating partial remainders that had to be ungrouped and regrouped. In Fig. 25.13, he made another subtle creative revision that remained consistent with his earlier work and experiences. When he was asked to explain his division method, Mark claimed that "it's like how we do adding and subtracting with regrouping, we're just doing it with division". Mark's manipulative abductive processing for division involving whole numbers necessitated a dynamic experience in which " a first *rough* and concrete experience" [25.11, p. 519] of the process enabled him to eventually develop a version of the long division process that "unfolded in real time" via thinking through doing.

25.3.2 Abduction in Mathematical Relationships

A study by *Arzarello* and *Sabena* [25.38] illustrates the important role of abduction in constructing mathematical relationships involving different signs. Signs pertain to the triad of signifier, signified, and an individual learner's mental construct that enables the linking between signifier and signified possible. Arzarello and Sabena underscore their students' use of semiotic and theoretic control when they argued and proved statements in mathematics. Semiotic control involves choosing and implementing particular semiotic resources (e.g., graphs, tables, equations, etc.) when they manipulate and interpret signs (i. e., type-1 semiotic action), while theoretic control involves choosing and implementing appropriate theories (e.g., Euclidean theorems) or parts of those theories and related conceptions when they "elaborate an argument or a proof" (i. e., type-3 semiotic action; [25.38, p. 191]). Between type-1 and

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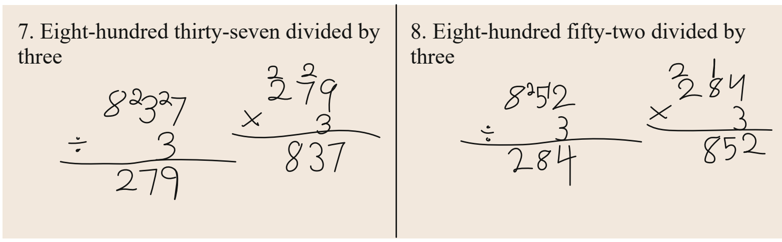

Fig. 25.12 Mark's initial numerical division processing

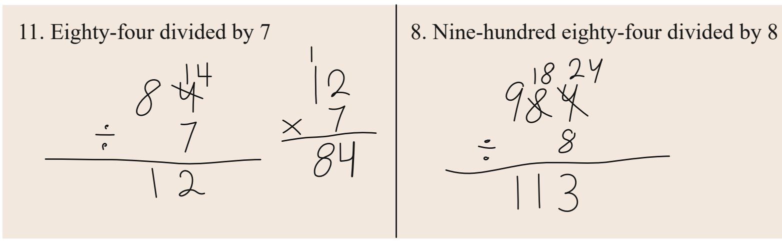

Fig. 25.13 Mark's manipulative abductive processing of the numerical methods shown in Figs. 25.11 and 25.12

type-3 semiotic action is a type-2 semiotic action that involves using abduction to identify relationships between signs and assessing the arguments. Based on their qualitative work with Grade 9 students, such [25.38, p. 202]

> "relationships between signs are examined and checked with redundant local arguments, and (economic, explanatory, and testable) hypotheses are detected and made explicit by means of abductions."

Furthermore, they note how [25.38, p. 204]:

> "abduction has an important role at this point. There is an evolution from a phase where the attention is mainly on the given signs, towards a phase where the logical-theoretical organization of the argument becomes the center of the activities and evolves from abductive to deductive and more formal structures. [...] Such an evolution implies a passage from actions of type 1 to actions of type 2 and then 3, and a shift of control by the student, i. e., passing from actions guided by semiotic control to actions guided by theoretical control. [...] Passing from type 1- to type 3-semiotic actions means an evolution from the data to the truth because of theoretical reasons. It is exactly this distinction that makes the difference between [...] a *substantial argument* and an *analytical argument*, which is a mathematical proof."

Arzarello and Sabena's study foregrounds the role of abduction in inferential processing and documents how a shift from abduction to deduction is likely to occur when students' mathematical thinking shifts in focus from the semiotic to theoretical, respectively. Studies by *Pedemonte* and colleagues [25.9, 36, 39] and *Boero* and colleagues [25.40, 41] also note the same findings in both algebra and geometry contexts. Across such studies we note how abduction is conceptualized in terms of its complex relationships with induction and deduction. Other studies do not deliberately focus on such shifts and relationships, making it difficult for students to see the value of engaging in abductive processing in the first place. For example, *Watson* and *Shipman* [25.42] documented the classroom event that happened in a Year 9 class of 13–14 year-old students in the UK that investigated the following task: Find a way to multiply pairs of numbers of the form $a + \sqrt{b}$ that results in integer products. While the emphasis of their study focused on learning through exemplification by using special examples to help students develop meaningful plausible structures, it seems that the abductive process for them became a matter of conjecturing relationships based on their experiences with their constructed examples. But certainly there is more to abductive processing than merely generating conjectures, as follows.

Several studies have suggested inferential model systems that show relationships between and among abduction, induction, and deduction. *Addis* and *Gooding* [25.10], for example, illustrate how the iterative cycle of

> abduction (generation) → deduction (prediction) → induction (validation) → abduction

might work in the formation of consensus from beliefs. *Radford*'s [25.43] architecture of algebraic pattern generalizations emphasizes a tight link between abduction and deduction, that is, hypothetico-deduction, in the fol-

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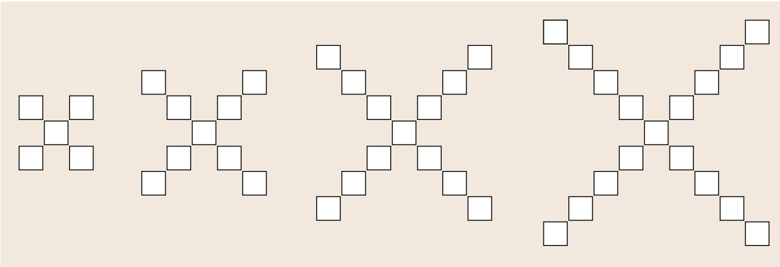

Fig. 25.14 Cross-squares pattern

lowing manner:

abduction (from particulars $p_1, p_2, \ldots, p_k$
to noticing a commonality C)
$\to$ transforming the abduction
(from noticing C to making C a hypothesis)
$\to$ deduction (from hypothesis C
to producing the expression of p_n) .

The studies conducted by *Rivera* [25.44] with groups of elementary (i. e., first through third grade) and middle school (i. e., sixth through eighth grade) students in the US on similar pattern generalization tasks capture two different inferential structures. Prior to a teaching intervention that involved using multiplicative thinking to establish pattern rules, both elementary and middle school student groups employed the same inferential structure of

abduction $\to$ induction $\to$ deduction
$\to$ deductive closure

that enabled them to generalize (correctly and incorrectly). The abductive phase in such a structure tended to be instinctual and iconic- or perceptual-driven. After the teaching intervention, however, Rivera observed that while the elementary student groups continued to model the same inferential structure in pattern generalizing, the middle school student groups skipped the induction phase and instead exhibited the following structure:

abduction and deduction $\to$ deductive closure .

Abduction in this phase was combined with deduction and thus became structured and inferential as a consequence of their ability to express generalizations in multiplicative form. For example, when the pattern generalization task shown in Fig. 25.14 was presented to both elementary and middle school student groups after the teaching experiment, sixth-grade student Tamara initially abduced the recursive relation +4, which enabled her to deduce the explicit rule $s = n \times 4 + 1$. She then used her combined abductive-deductive inference to perform deductive closure, in which case she induced the given stages and predicted the correct number of squares for any stage in her pattern. Tamara's empirical justification of her explicit rule for the total number of squares s involves seeing a fixed square and four copies of the same *leg* that grew according to the stage number n. In the case of third-grade Anna, her multiplicative-driven abductive processing enabled her to both construct and justify the same explicit rule that Tamara established for the pattern. However, she needed to express her answers inductively, as follows

$$4 \times 1 + 1,\ 4 \times 2 + 1,\ 4 \times 3 + 1,\ \ldots, 4 \times 100 + 1,\ \ldots$$

25.4 Enacting Abductive Action in Mathematical Contexts

We close this chapter by providing four suggestions for assisting students to enact meaningful, structured, and productive abduction action. Together the suggestions target central features in abductive cognition, that is, thinking, reasoning, processing, and disposition. Empirical research in mathematics education along these features is needed to fully assess the extent and impact of their power in shaping mathematical knowledge construction.

25.4.1 Cultivate Abductively-Infused Guesses with Deduction

Students will benefit from knowing how to generate *new* guesses and conjectures that can explain a problem and occur "within the wider scope of the process of inquiry" [25.20, p. 116]. That is, while abductions certainly emerge from perceptual judgments, in actual practice the more useful ones are usually constrained

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and logical as a consequence of knowing the problem context and being "compounds of deductions from general rules" (i. e., hypothetico-deductivist) that individual knowers are already familiar with (Peirce, quoted in [25.20, p. 119]). *Tschaepe* writes, "(w)e guess in an attempt to address the surprising phenomenon that has led to doubt; it is our inchoate attempt to provide an explanation" [25.20, p. 118]. Viewed in this sense [25.20, p. 122],

> "[a]bduction is a logical operation, and guess is logical insofar as it is a type of reasoning by which an explanation of a surprising phenomenon is first created, selected, or dismissed [...] Guessing is the creative component of abductive inference in which a new idea is first suggested through reasoning."

25.4.2 Support Logically-Good Abductive Reasoning

Students will benefit from knowing how to develop abductions that are logically good, that is, they are: clear (i. e., can be confirmed or discomfirmed); can explain the facts; are capable of being tested and verified; and can lead to true explanations that establish "sustainable belief-habits" [25.2, p. 163]. Such explanations may be new and may emerge from guesses and instincts, but, Khachab writes [25.2, pp. 171–172],

> "logical goodness is the reason for *abduction*, under its diverse meanings. No matter *how* abduction *actually* generates *new* ideas – whether it is abductive inference, strategic inference, instinctive insight, etc. – its purpose is, ultimately, to provide true explanatory hypotheses for inquiry. And, in this regard, *new* hypotheses should always be evaluated in reference to their goodness."

25.4.3 Foster the Development of Strategic Rules in Abductive Processing

Paavola [25.12] distinguishes between definitory and strategic rules. While definitory rules focus on logic and logical relationships, strategic rules pertain to "goal-directed activity, where the ability to anticipate things, and to assess or choose between different possibilities, are important" [25.12, p. 270]. Thus, abductive strategies produce justifications for given explanatory hypotheses, including justifications for "why there cannot be any further explanation" [25.12, p. 271]. Hence, all generated abductive inferences conveyed in the form of discoveries provide an analysis or explanation of the underlying conceptual issues and are not merely reflective of mechanical recipes or algorithms for generating ideas and discoveries. Furthermore, the analysis or explanation should present "a viable way of solving a particular problem and that it works more generally (and not only in relationship to one, particular anomalous phenomenon)" [25.12, p. 273] and fit the "constraints and clues that are involved in the problem situation in question" [25.12, p. 274].

25.4.4 Encourage an Abductive Knowledge-Seeking Disposition

Sintonen's [25.45] interrogative model of inquiry that employs an explicit logic of questions demonstrates the significance of using certain strategic principles and why-questions as starting points in abductive processing. Questions as well as answers drive discoveries and the scientific process. Questions, especially, "pick out something salient that requires special attention, and that it also gives heuristic power and guidance in the search for answers" [25.45, p. 250]. Furthermore, [25.45, p. 263],

> "principal questions are often explanation-seeking in nature and arise when an agent tries to fit new phenomena to his or her already existing knowledge. Advancement of inquiry can be captured by examining a chain of questions generated. By finding answers to subordinate questions, an agent approaches step by step toward answering the big initial question, and thus changes his or her epistemic situation."

Students will benefit from situations and circumstances that engage them in a knowledge-seeking game in which they "subject a source of information [...] to a series of strategically organized questions. This Sherlock Holmes method therefore is at the heart of abductive reasoning" [25.45, p. 254]. Furthermore, the interrogative model allows conclusions (i. e., answers) to emerge. "For abductive tasks", Sintonen writes [25.45, p. 256],

> "the goal must be understanding and not just knowledge. A rational inquirer who wants to know why and not only that something is the case must, after hearing the answer, be in the position to say *Now I know (or rather understand) why the (singular or general) fact obtains*. Obviously this condition is fulfilled only if she or he knows enough of the background to able to insert the offered piece of information into a coherent explanatory account."

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Acknowledgments. The research that is reported in this chapter has been funded by the National Science Foundation under Grant Number DRL 0448649. All the views and opinions expressed in this report are solely the author's responsibility and do not necessarily reflect the views of the foundation.

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