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Visual templates in pattern generalization activity

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Abstract In this research article, I present evidence of the existence of visual templates in pattern generalization activity. Such templates initially emerged from a 3-week design-driven classroom teaching experiment on pattern generalization involving linear figural patterns and were assessed for existence in a clinical interview that was conducted four and a half months after the teaching experiment using three tasks (one ambiguous, two well defined). Drawing on the clinical interviews conducted with 11 seventh- and eighth-grade students, I discuss how their visual templates have spawned at least six types of algebraic generalizations. A visual template model is also presented that illustrates the distributed and a dynamically embedded nature of pattern generalization involving the following factors: pattern goodness effect; knowledge/action effects; and the triad of stage-driven grouping, structural unit, and analogy.

Keywords Pattern generalization · Algebraic thinking · Visual templates · Additive reasoning · Multiplicative reasoning · Algebraic generalization · Abduction · Pattern goodness

1 Introduction

Figural patterns, whether constructed ambiguously (see, e.g., Fig. 1) or in a well-defined manner (see, e.g., Figs. 2 and 3), consist of stages whose parts could be interpreted as being

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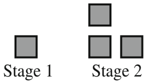
Below are the first two stages in a growing pattern of squares.



1. Continue the pattern until stage 5.

2. Find a direct formula in two different ways. Justify each formula.

3. If none of your formulas above involve taking into account overlaps, find a direct formula that takes into account overlaps. Justify your formula.

4. How do you know for sure that your pattern will continue that way and not some other way?

5. Find a different way of continuing the pattern and obtain a direct formula for this pattern.

Fig. 1 Square pattern task in compressed form

configured in a certain way.[1] The primary task for learners is to engage in meaningful and mathematically viable *pattern generalization*, which involves coordinating their perceptual[2] and symbolic inferential abilities so they are able to *construct* and *justify* a plausible and algebraically useful structure[3] that could be conveyed in the form of a direct formula (i.e., a function-based equation in explicit form; cf. Lee, 1996).

Meaningful patterns are structured sequences of objects. If we rearrange Fig. 1 so that the stages look like the one shown in Fig. 4, then the task of constructing an algebraic generalization for learners might become either unnecessarily complicated or a meaningless activity. The point is, whether stages in a pattern are ambiguous or well defined, learners should be able to establish a stable relationship within and across stages that would enable them to perform extensions accordingly and in a mathematically consistent manner.

Because pattern construction is a subjective and constructive activity, learners need to know how to coordinate their perceptual and symbolic inferences effectively in relation to

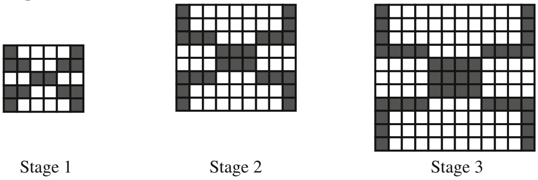
[1] I prefer to use the term figural pattern in order to convey what I assume to be the "simultaneously conceptual and figural" (Fischbein, 1993, p. 160) nature of mathematical patterns. The term "geometric patterns" is not appropriate due to a potential confusion with geometric sequences (as instances of exponential functions in discrete mathematics). Also, I was not keen in using the term "pictorial patterns" due to the (Peircean) fact that figural patterns are not mere pictures of objects but exhibit characteristics associated with diagrammatic representations. The term "ambiguous" shares Neisser's (1976) general notion of ambiguous pictures as conveying the "possibility of alternative perceptions" (p. 50).

[2] Especially in the case of patterning tasks that involve figural cues, among the most important perception types that matter involve visual perception. Visual perception involves the act of coming to see; it is further characterized to be of two types, namely, sensory perception and cognitive perception (Dretske, 1990). Sensory (or object) perception is when individuals see an object as being a mere object in itself. Cognitive perception goes beyond the sensory when individuals see or recognize a fact or a property in relation to the object. For example, students who see consecutive groups of figural stages in Fig. 7 as mere sets of objects exhibit sensory perception. However, when they recognize that the stages taken together actually form a pattern sequence of objects, they manifest cognitive perception. Cognitive perception necessitates the use of conceptual and other cognitive-related processes, enabling learners to articulate what they choose to recognize as being a fact or a property of a target object. It is mediated in some way through other types of visual knowledge that bear on the object, and such types could be either cognitive or sensory in nature.

[3] I share Resnik's (1997) view that the "primary subject-matter [of mathematics] is not the individual mathematical objects but rather the structures in which they are arranged" (p. 201). Pattern stages basically convey positions in some overall structural relationship and, so, do not have an identity or distinguishing features outside that relationship despite the fact that *specializing* on a particular stage, which is considered a useful generalizing strategy, may give the impression of a structure.

Fig. 2 Square frog pattern task in compressed form

Consider the pattern below.



1. How does Stage 4 look like? Either describe it or draw it on a graphing paper.

2. Find a direct formula for the total number of gray square tiles at any stage. Explain your formula.

3. How many gray square tiles are there in stage 11? How do you know?

4. Which stage number contains a total of 56 gray square tiles Explain.

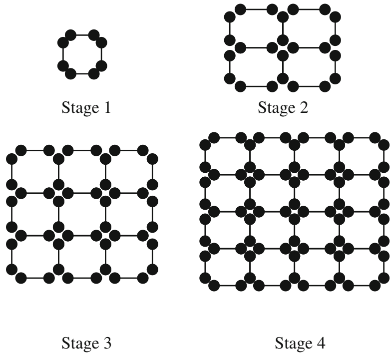
an interpreted structure of a pattern that applies to both the known and unknown stages. On the basis of recent findings at the K-9 and preservice elementary undergraduate levels involving generalization in patterning activity, we still need further research in terms of how such coordination could take place more effectively in a systematic manner (Mulligan & Mitchelmore, 2009; Warren & Cooper, 2008; Rivera & Becker, 2008a; Rivera, 2007; Billings, Tiedt, & Slater, 2007; Lannin, Barker, & Townsend, 2006; Lannin, 2005; Steele & Johanning, 2004; Stacey & MacGregor, 2001). Recent patterning studies that work within a constructivist framework have significantly contributed to our developing understanding of how elementary and middle-school students obtain algebraic generalizations involving figural and numerical patterns. In this article, I qualitatively assess the implications of a design research experiment[4] on pattern generalization that emphasize visualization and multiplicative reasoning at the middle-school level.

I resolve the following two related research questions: How do middle-school students perform pattern generalization that leads to an algebraically useful structure for a given figural pattern on the basis of a few initial stages? Also, considering their developmental capacities for perceptual and symbolic inference and their relatively few experiences in structure formation and development, how do they impose a structure on sets of ambiguous and well-defined stages such as the ones shown in Figs. 1 and 2, respectively, so that what evolves is a mathematically acceptable pattern generalization? One resolution empirically routes us to an elaboration of a cognitive model that accounts for middle-school students' algebraic generalizations on the basis of findings and observations drawn from both the clinical interviews and the design experiment.

[4] Intrinsic to classroom teaching experiments that employ design research are two objectives, that is, developing an instructional framework that allows specific types of learning to materialize and analyzing the nature and content of such learning types within the articulated framework. Thus, in every design study, theory and practice are viewed as being equally important, which includes rigorously developing and empirically justifying a domain-specific instructional theory relevant to a concept being investigated. Further, the content of the proposed instructional theory involves a well-investigated learning trajectory and appropriate instructional tools that enable student learning to take place in various phases of the trajectory. Finally, instruction in design studies is characterized as having the following elements: it is experientially real for students; it enables students to reinvent mathematics through, at least initially, their commonsense experiences; and it fosters the emergence, development, and progressive evolution of student-generated models.

Fig. 3 The square array pattern in compressed form

Consider the following array of sticks below.



A. Find a direct formula for the total number of sticks at any stage in the pattern. Justify your formula.

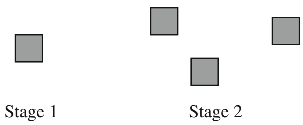
B. Find a direct formula for the total number of points at any stage in the pattern. Justify your formula.

In this article, I discuss visual templates in relation to pattern generalization activity among a group of seventh- and eighth-grade students (ages 12 and 13 years) in an urban school in Northern California. In particular, I focus on three template types, as follows: additive, multiplicative, and pragmatic. The pattern generalization tasks that the students pursued involved one semifree construction (Fig. 1) and two well-defined nonlinear patterns (Figs. 2 and 3). In the next two sections, I further clarify the notion of meaningful pattern generalization and discuss in some detail the nature and types of visual templates on the basis of theoretical perspectives and empirical findings I have drawn from the relevant literature.

2 Context of meaningful pattern generalization in this study

Meaningful pattern generalization involves the coordination of two interdependent actions, as follows: (1) *abductive–inductive action on objects*, which involves employing different ways of counting and structuring discrete objects or parts in a pattern in an algebraically useful manner; and (2) *symbolic action*, which involves translating (1) in the form of an algebraic generalization. The idea behind *abductive–inductive action* is illustrated in Fig. 5. It is an empirically verified diagram of phases in pattern generalization that I have drawn from a cohort of sixth-grade students who participated in a constructivist-driven pattern generalization study for two consecutive years (Rivera, 2008; Rivera & Becker, 2008a, b, 2009a, b).

Fig. 4 Reconfigured square pattern



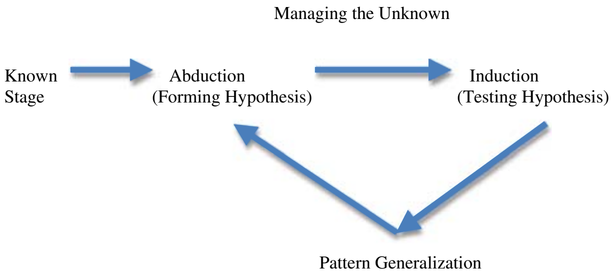


Fig. 5 Abductive–inductive action involving pattern generalization

As shown in Fig. 5, abduction is situated at the kernel of the generalizing process as students begin to explore a plausible rule that can both explain the known stages and be used to construct the unknown stages in any given pattern with incomplete instances. The abductive phase is when students begin to offer an explanatory hypothesis for a given pattern on the basis of the available instances, which is then used to extend the pattern (*near generalization tasks* such as determining the total number of sticks in the fourth and fifth stage of the pattern in Fig. 3) and repeatedly tested—that is, the inductive phase—leading to either confirming a rule (preferably a direct formula) or further developing another abduction. When the rule is confirmed, a generalization emerges that consequently enables students to deal with *far generalization tasks* such as obtaining the total number of sticks in the 77th stage of the pattern in Fig. 3 without laboriously constructing the preceding 76 stages. A further abduction is warranted when the rule makes it almost difficult to deal successfully with a far generalization task.

3 Nature, characteristics, and types of visual templates in relation to this study

3.1 Nature of visual templates

In the case of structured sets, that is, those sets with "specific relations, functions, or constants" (Giaquinto, 2007, p. 214) such as the set of natural numbers with operations of addition and multiplication, the successor property, and the constant 0, Giaquinto (2007) claims that structure discernment takes shape either through theory association or with the use of a *visual* or *perceptible template*. When a structured set is either a particular instantiation of an established theory or shares an isomorphic structure with some known theory, individuals can deal with the elements in the set in the same manner that they would manipulate the elements in the established theory. Further, when they do not have access to such a theory, their visual powers can help them discover and explain new relationships. Giaquinto (2007) articulates this visual activity clearly as follows: "We can know them [i.e., structures] through visual experience of instances of them, somewhat as we can know the butterfly shape from seeing butterflies" (p. 236). However, such visual experience seems to presume access to a sophisticated mechanism that assists in establishing an isomorphic relationship between the concrete and abstract representation of an intended structure.

In this article, I discuss the cognitive implications of a pattern generalization scheme that I refer to as a *visual template*, a term that I appropriate from Giaquinto (2007) who used it in the context of mathematical objects. But the manner in which I appropriate it is more general than Giaquinto's (2007) and shares more of the characteristics identified by Neisser (1976) who explored the notion of *template matching* in the context of pattern recognition of everyday objects. Giaquinto (2007), on the one hand, developed his notion of a visual template from Resnik (1997) who describes a template as a "concrete device for representing how things are shaped, structured, or designed" (Resnik, 1997, p. 227). When we manipulate a template, much as we use blueprints, somehow the concrete experience should provide either a structurally similar or structurally contained context that enable us to make sense of the corresponding abstract patterns and their properties. For example, when second-grade children begin to group concrete objects in a particular way, then the concrete understanding that comes with the grouping experience should somehow foreshadow what eventually would be known as the general structure of place value notation or counting systems. There is, thus, an expectation of a percept-to-concept projection process that is involved with the visual component assisting in constructing or contextualizing the abstract elements involved. Neisser (1976), on the other hand, articulates the important role of prototypes or canonical forms as a standard or basic model that enable children to either learn a characteristic of a new object or compare the new with an existing object.

The visual aspect of such templates emphasizes the role of the "'eye' as a legitimate organ of discovery and inference," thus, "restor[ing the] balance between mathematical methodologies" (Davis, 1993, p. 342). Davis (1993) astutely points out how "discovery is not usually made in the deductive way" (p. 342), that is, not in the sole context of logical inference and verbal deduction but through our visual capacities of observing and seeing. Also, following Arcavi's sense of visualization in mathematics (2003), visual templates as a type of visual strategy enable students to *see the unseen* of an *abstract world* that is dominated by relationships and conceptual structures that are not always directly evident. The characterization offered by Zazkis, Dubinsky, and Dautermann (1996) is also worth noting. Within the context of constructive action, they see visualization as conveying a relationship "between an internal construct and something to which access is gained through the senses" (p. 1996). However, in this article, I characterize visual action in a much stronger sense, that is, within the context of visual representations that employ "visuospatial relations in *making inferences* about corresponding conceptual relations" (italics added; Gattis & Holyoak, 1996, p. 231).

3.2 Characteristics of visual templates

Hutchins's (1995) notion of *strips* that novices and experts use to learn rules in navigation schools describes an important fundamental characteristic of visual templates. In navigation schools, trainees learn to properly implement written sequences of actions through procedures or strips that guide them toward the correct solutions. A strip is a written form consisting of blank spaces that trainees fill in the correct order. While strips as a type of sequential knowledge appear on the surface to be leaving little to no room in helping trainees develop conceptual understanding (since they basically follow them faithfully like a strict script), Hutchins (1995) notes that they depend on how trainees develop them in memory. Remembering and memorizing strips could not be one and the same thing since the experience of remembering could take many forms. For example, experienced pilots who inspect airplanes oftentimes engage in an attention flow as a consequence of having developed implicit spatial or sequential relations that they initially acquired from a strip.

Visual templates generally operate like Davis's (1984) *visual-mode-rated sequence* (VMS), that is, an everyday coping mechanism in which a knower uses a sequence of key markings (say, landmarks) to achieve his or her desired goal (say, destination). Such templates

> function as a tool to extricate oneself from situations in which one may be uncertain about how to proceed. As such it is linked, in this case, not so much to concepts and ideas, but rather to perceptions that lead procedural decisions. (Arcavi, 2003, p. 224)

In VMS, one begins with a visual clue V_1 that elicits a procedure P_1 that then elicits another visual clue V_2 leading to a procedure P_2, etc.

Visual templates also help learners engage in meaningful and purposeful *diagram parsing* (cf. Koedinger & Anderson, 1995). Expert geometry provers use diagram parsing or perceptual chunking that involves identifying basic (i.e., low-level components) and advanced (i.e., high-level inductive components) configurations in a diagram, which then enable them to instantiate the corresponding concepts or processes leading to steps in a plausible deductive proof.

3.3 Types of visual templates: additive, multiplicative, and pragmatic schemes

Figure 6 illustrates the fundamental difference between additive and multiplicative schemes involving whole-number objects in terms of level of abstraction and the complexity of inclusion relations involved. Drawing on Piaget's (1987) conceptualization, Clark and Kamii (1996) share the same view that multiplication is an "operation that requires higher-order multiplicative thinking that children construct out of their ability to think additively" (p. 42). They note that while addition "is inherent in the construction of number, which is accomplished by the repeated addition of ones," multiplication "is a more complex operation that is constructed out of addition at a higher level of abstraction" (ibid.). As illustrated in Fig. 6, an additive scheme necessitates only one level of abstraction and one inclusion relationship (e.g., each unit of three consists of ones; one gets included in two, two in three, three in four, etc.). In the case of multiplicative schemes, there is a need to establish *simultaneously* (as indicated by the arrows) the following levels of abstraction and inclusion relationships: a many-to-one mapping (e.g., between three units of one and one unit of three) and at least two levels of relationships (e.g., horizontally by ones: one in two, two in three, etc.; horizontally by threes: one three in two threes, 2 threes in 3 threes; etc.; vertically by threes: one threes, two threes, three threes, etc.).

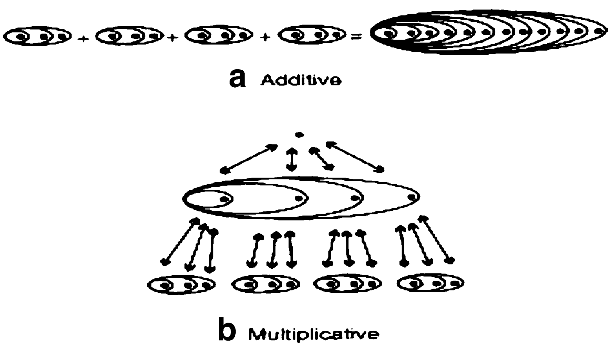


Fig. 6 Comparing **a** additive and **b** multiplicative thinking (Clark & Kamii, 1996, p. 42)

Two important points concerning the *notion of a unit* are worth noting in light of its central role in the construction of additive and multiplicative schemes. The *first* point concerns matters involving counting. According to Sophian (2007), counting involves "the choice of a unit.... In principle, provided we are consistent about what we take as a unit, we can count any sort of discriminable element in any kind of array" (p. 64). Further, she astutely emphasizes how

> mathematical thinking requires a concept of *unit* that goes beyond our everyday notions of objects and groups of objects. Specifically, it requires flexibility in the choice of units, together with a concern for equivalences among units..... Notably, in counting, *it is the identity of the unit rather than its size that matters*: Some of the shoes may be baby shoes and others large men's shoes, but we treat each of them as one shoe (or one half of a pair of shoes). (italics added for emphasis; Sophian, 2007, p. 66)

In an additive scheme, which involves a single-level abstraction, we employ a unit that may or may not be independent of the quantities involved in addition (and subtraction). For example, in the comparison situation, "Maria is two inches taller than her sister Jana," the unit—inches—pertains to height that conveys a linear measure. We could, of course, use other linear units such as centimeters. In the combination situation, "There are 29 students in a room with 12 boys and 17 girls," the unit—number of people—is independent of either quantity being added. In a multiplicative scheme, which involves multilevel abstractions, we employ a unit that is actually drawn from one of the quantities involved in multiplication (and division). For example, in the comparison situation, "Maria is twice as tall as her sister Jana," the unit is Jana's height which we then use to compare Maria's height with. In summary, both types of schemes necessitate the use of a *common unit* that further enables us to understand the nature of the inclusion relationships shown in Fig. 6. In Fig. 6a, the common unit may or may not be independent of the quantities involved. In Fig. 6b, the many-to-one map initially relies on the choice of a common unit that is then used to systematically count the quantities involved.

The *second* point involves the implications of different ways of seeing a unit in relation to the construction of perceptual and symbolic inferences. In figural pattern generalizing, in particular, I discuss briefly for now the well-established *Law of Good Gestalt* (Metzger, 2006, pp. 19–27). Later, in this article, I use the terms *pattern goodness* and *Gestalt effect* interchangeably and they both refer to the same Gestalt law. This law pertains to the order in which we perceive, discern, and organize a figure, a picture, or an image—that is, we are predisposed to organizing on the basis of what naturally belongs or fits together, including that which is simple and recognizable enough that enables us to associate or specify a geometric shape or algebraic formula. Hence, a figural pattern that is *high in Gestalt goodness* tends to have an interpreted structure that reflects the orderly, balanced, and harmonious form of the pattern, which allows learners to specify an algebraically useful formula easily. A figural pattern that is *low in Gestalt goodness* is interpreted as being disorganized with a complex (unbalanced) structure that either has no easily discernible parts or consists of parts that have no "natural divisions," which makes the task of constructing an algebraically useful formula difficult to accomplish. In both cases of pattern goodness, the initial choice of a common unit plays a significant role in the construction of an algebraic generalization, which eventually takes an additive or a multiplicative form depending on whether a student is thinking additively or multiplicatively, respectively, about the unit and its relationship to the overall interpreted structure of the pattern.

Table 1 provides a summary of at least seven different types of additive- and multiplicative-based algebraic generalizations that students have been known to produce in relation to pattern generalization activity. They have been drawn from recent studies involving figural patterns that demonstrate the remarkable, albeit fundamental, view that individuals tend to see patterns differently depending on how they conceptualize their units (see the ZDM issue on pattern generalization for references to most studies; Rivera & Becker, 2008a). More frequently than the other types, students tend to develop *constructive standard and nonstandard generalizations*. The *constructive* dimension refers to the fact that an interpreted structure relative to some pattern is seen as consisting of *nonoverlapping parts* that when added together form the perceived shape that applies across the stages in the pattern. The term *standard* and *nonstandard* refer to the algebraic terms in a direct expression, that is, standard means the terms are already in simplified form while nonstandard contains terms that can still be further simplified. Constructive generalizations reflect the use of either an additive or a multiplicative scheme. Although not as frequent as constructive generalizations, students also develop *deconstructive generalizations*, that is, they see the known figural stages in a pattern as consisting of *overlapping parts* that can be decomposed quite conveniently. In very few cases, students engage in *auxiliary-driven constructive or deconstructive generalizations*, that is, they see each known figural stage in a pattern in the context of a larger configuration that has a well-known and/or simpler structure. Introducing an auxiliary set of objects strategically enables them to better see the larger configuration in order to obtain an appropriate generalization for the pattern rather quickly and easily. Very few students also have been observed to engage in *transformation-based generalizations*, which initially involves actions of moving, reorganizing, and transforming parts in a figural stage of a pattern into some recognizable figure with a more familiar structure.

A pragmatic scheme occurs in problem situations when a student combines additive and multiplicative schemes. This type of scheme shares Krutetskii's (1976) notion of harmonic style in his characterization of different cognitive styles in mathematical problem solving (the two other styles being visual and verbal/analytic). In relation to patterning activity, the clinical interviews that I conducted with 22 beginning ninth-grade students and 42 preservice elementary undergraduate majors indicate a disposition among several students toward using a combination of visual and numerical strategies (Rivera & Becker, 2007; Becker & Rivera, 2005).

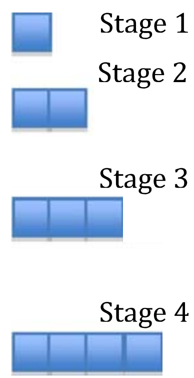
4 Method

In this section, I talk about methodological issues relevant to documenting and establishing the existence of visual templates in pattern generalization activity. I begin with the surrounding context of the research study in order to understand how I arrived at such templates.

4.1 Participants in the design experiment

The eighth-grade Algebra 1 class consists of 34 seventh- and eighth-grade students (28 of Southeast Asian origins; two African–Americans; one Hispanic; two South Asians; one Caucasian) in Northern California who participated in a 10-month (Fall 2007 to Spring 2008) research investigation on generalization and abstraction at the middle-grade level. In this urban class that I taught throughout the school year, there were 12 seventh graders

Table 1 Types of algebraic generalizations

Type of algebraic generalizations	Figural characteristics (perceptual inference)	Algebraic formula (symbolic inference)	Examples
Additive constructive standard	Seeing figural patterns as consisting of *nonoverlapping* parts	The terms in the formula are in simplified form This is the simplest case that applies to all linear figural patterns of the type $y = x + b$ (i.e., the constant addition of 1 object from stage to stage)	
Additive constructive nonstandard	Seeing figural patterns as consisting of *nonoverlapping* parts	The terms in the formula are in expanded, nonsimplified form	Karen's formula in relation to Fig. 9: $x = n - 1 + n - 1 + 1$
Multiplicative constructive standard	Seeing figural patterns as consisting of *nonoverlapping* parts	The terms in the formula are in simplified form	Jamal's formula in relation to Fig. 7: $C = 3n + 1$

Multiplicative constructive nonstandard	Seeing figural patterns as consisting of *nonoverlapping* parts	The terms in the formula are in expanded, nonsimplified form	Ollie's formula in relation to Fig. 7: $C = 2n + (n + 1)$ Emma's formula in relation to Fig. 7: $C = 2(n + 1) + (n - 1)$
Deconstructive	Seeing figural patterns as consisting of *overlapping* parts. Hence, counting involves taking into account multiple counts that can be determined when the parts are appropriately decomposed	The terms in the formula could be standard or nonstandard in form The terms could convey the use of either an additive or a multiplicative scheme	Tamara's formula in relation to Fig. 7: $C = 3(p + 1) - 2$
Auxiliary-driven constructive or deconstructive	Seeing figural patterns as parts of a larger configuration that has a well-known and/or simpler structure. The introduction of an auxiliary set enables learners to see the larger configuration more easily	The terms in the formula could be standard or nonstandard in form The terms could convey the use of either an additive or a multiplicative scheme	Diana's formula in relation to Fig. 9 as a result of introducing an auxiliary set (shown in Fig. 11): $n = x^2 - (x - 1)^2$
Transformation-based constructive or deconstructive	Seeing figural patterns in a different way by initially moving, reorganizing, and transforming a figural stage into some recognizable figure that has a structure that is familiar to the learner	The terms in the formula could be standard or nonstandard in form The terms could convey the use of either an additive or a multiplicative scheme	See ESM 11 for a figural transformation of Tamara's Fig. 16 pattern that leads to the formula: $s = n(n - 1) + 1$

(mean age of 12; eight females, four males) and 22 eighth graders (mean age of 13; 14 females; eight males). I should note that prior to this particular yearlong research, 15 eighth graders ("cohort 1") from the above group of 34 students participated in an earlier 2-year research investigation on pattern generalization. Cohort 2 consists of the remaining 19 seventh- and eighth-grade students who joined the Algebra 1 class.

4.2 An initial pattern generalization approach to the grade 8 algebra curriculum

While the California content standards in Algebra 1 do not include patterns in the list of required topics, in my class, I used linear pattern generalization activity to help students develop their initial understanding of functions, domain and range, linear functions, equations, equivalence, and graphs. The students explored linear pattern generalization tasks in the context of a 3-week design-driven classroom teaching experiment. (See Rivera and Becker (2009b) for details regarding the teaching experiment.)

I note that prior to the above teaching experiment on pattern generalization, the students participated in an earlier teaching experiment on integer and polynomial operations. In this experiment, they used algeblocks (i.e., algebra tiles in three dimensions) in developing the notion of a *unit*. For example, in adding (and subtracting) expressions such as $3x^2+5x^2$, the notion of like or similar terms emerged as a consequence of recognizing a common unit (i.e., an expression, say, x^2), which explained to them why the numerical coefficients and not the variable factors were the ones that necessitated combination by either addition or subtraction.

Considering the fact that there were two student cohorts with different prealgebraic experiences, the class explored the concept of multiplication involving two whole numbers or monomials in two ways, as follows: (1) within a set-theoretic view, which involves thinking and counting objects in terms of groups (e.g., $3 \cdot 2$ means three groups of two green cubes with the algeblocks; $4x^2$ means four groups of two yellow squares); and (2) within a geometric view, which involves seeing rows and columns of objects (e.g., $3 \cdot 2$ means three rows and two columns of green cubes). The set-theoretic view reinforces the notion of a common unit, while the geometric view is useful in making sense of multiplication and division involving polynomial expressions.

An interesting consequence of the above teaching experiment on integer and polynomial operations involves the manner in which the students approached linear pattern generalization activity. Their understanding of the salience of the unit concept enabled them to develop and gain a better understanding of equivalent direct expressions and formulas. They acknowledged the fact that their individual perceptual ability to interpret a structure of a pattern relied on what they considered to be their choice of a unit relative to the known pattern stages that would then be projected onto the extensions and the unknown stages (Rivera & Becker, 2009b). Figure 7 provides a sample of student work on a pattern generalization task. It illustrates the students' additive- and multiplicative-based direct formulas that they established and justified visually on the basis of what they individually interpreted to be a plausible structure of the figural pattern.

4.3 Participants, research design, and task analysis in this study

Eight eighth-grade students from cohort 1 and three seventh-grade students from cohort 2 (four males, seven females) each participated in a 55-min clinical interview that happened four and a half months after the teaching experiment on pattern generalization. All 11 students also participated in two clinical interviews immediately before and after the teaching experiment.

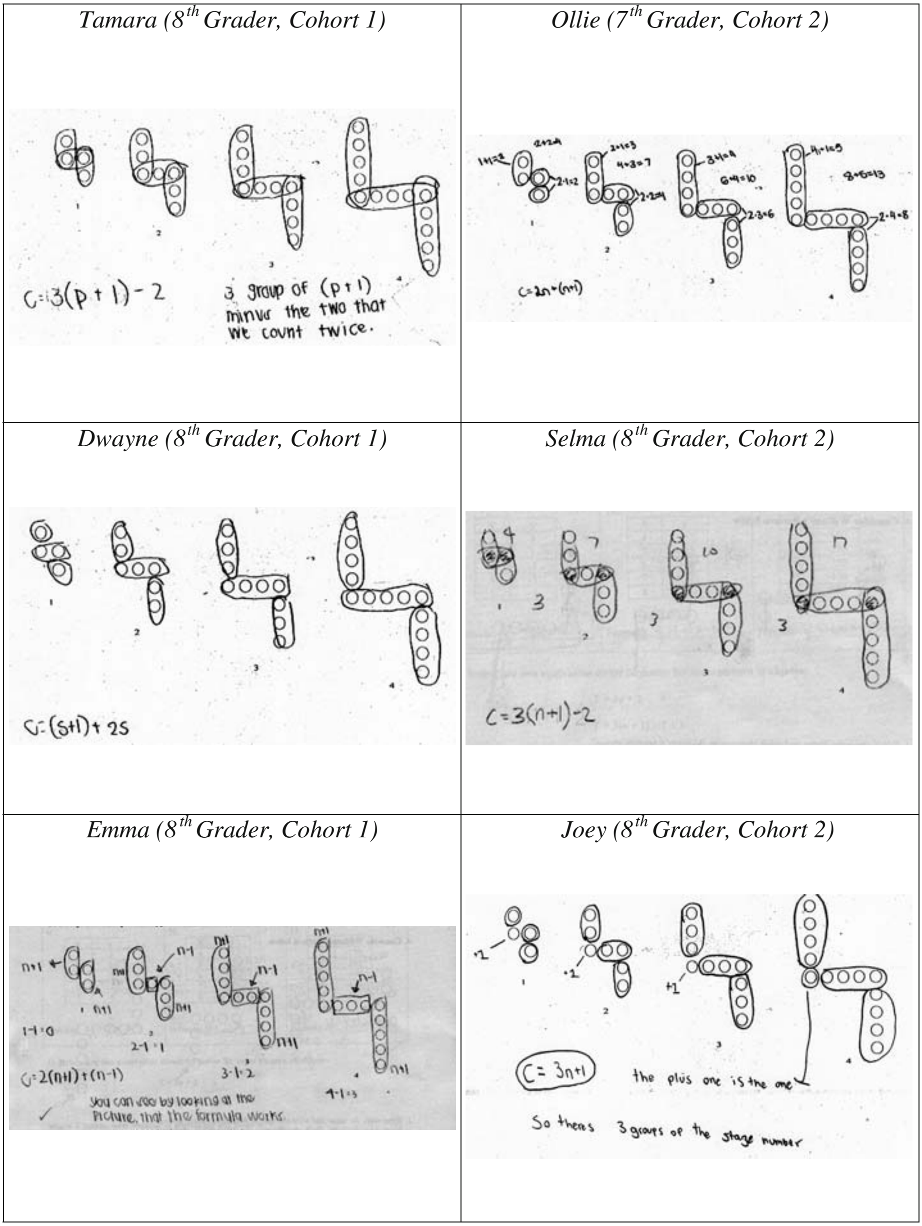


Fig. 7 Samples of students' visuo-alphanumeric generalizations in relation to the growing chair pattern

The overall research design could be described in diagram form as follows: X_1–O–X_2–X_3, where X_i refers to the *i*th clinical interview and O means teaching experiment intervention.

In establishing a case for the existence of visual templates in pattern generalization, I took into account the three methodological issues below.

First, I thought that each student would be able to articulate his or her visual template after a prolonged period of time in which no patterning activity was pursued on purpose.

While the clinical interviews immediately after the teaching experiment (X_2) gave some indication that such templates existed, however, I was interested in assessing their stability over time.

Second, since such visual templates target structure discernment and construction, the three tasks used in the X_3 clinical interview involved patterns that had ambiguous (Fig. 1) and well-defined stages (Figs. 2 and 3). Such nonroutine tasks were meant for them to come face to face with their *default* pattern generalization schemes, that is, visual templates.

The ambiguous task shown in Fig. 1 has been drawn from Dörfler's (2008) insights about the current state of research on patterns. He notes that patterns with well-defined stages impress on learners the view that "there is an expected direction of generalizing" that "intimate one and only way [of continuing] a figural sequence" that could result in "a strong regulating or even restrictive impact" on their thinking (p. 153). He then recommends a different way of asking students to think about (figural) patterns, as follows:

> How otherwise can one ask for, say, the number of matchsticks... in an "arbitrary" item of the sequence? The situation would presumably be much more open if one asked simply "How can you continue?" or "What can you change and vary in the given figures?"... I rather want to hint to possible further directions for research... a plea for "free" generalization tasks not restricted by pregiven purposes. (Dörfler, 2008, p. 153)

The Fig. 1 task targeted the students' ability to pattern generalize by initially drawing on a structure of their own making (with an initial constraint, hence, a semifree construction task), which includes assessing how their visual templates enabled them to accomplish the task.

The two well-defined tasks shown in Figs. 2 and 3 exemplify nonlinear algebraic generalizations. In the Algebra 1 class, due to content constraints in the state-recommended mathematics curriculum, we did not explore nonlinear pattern generalization activities. My basic goal in presenting them to the students was to assess the *representational power* of their visual templates in an unfamiliar nonroutine context, that is, in Berkeley's (2008) sense, the capacity of "any set of symbols [to] have the necessary flexibility to make manifest all the things that it needs to for the task at hand" (p. 97).

Third, I conducted all the clinical interviews.[5] Each participant worked on the three tasks (Figs. 1, 2, and 3) one at a time. He or she was asked to think aloud. Square blocks were provided in the case of the Fig. 1 task and a centimeter graphing paper in the case of the Fig. 2 task. In addition to the three tasks, I asked each student to explain what patterns meant to him or her, including factors that assisted him or her in constructing a direct formula (Fig. 8).

4.4 Data analysis

The data analysis process followed the steps described by Healy and Hoyles (1999). Individual case studies were constructed that consist of selected interview segment

[5] I am aware of issues surrounding researcher-driven interviews. The X_3 interviews that I conducted with the 11 students were third in a series of clinical interviews conducted during the 10-month project. A colleague who was external to the project conducted both X_1 and X_2 interviews. Also, the X_3 interview would in fact be seventh in the case of the cohort 1 participants. Yanos and Hopper (2008) suggest the use of repeated interviews as a way of minimizing what Bourdieu (1999) refers to as "false, collusive objectification" whereby interviewees respond in ways that please their interviewers. Further, the findings that I have drawn from the X_3 clinical interviews and that are reported in this article were triangulated with two other longitudinal data sources (i.e., available student written work and earlier clinical interviews conducted by a colleague).

Fig. 8 Perceptions of patterns and direct formula task in compressed form

Respond to the following two questions below.

1. How can you tell for sure when a group of instances (stage 1, stage 2, stage 3, etc.) forms a pattern?

2. How did you come to understand how the direct formulas for patterns are constructed?

transcripts, written work on pattern tasks, and data analysis of the relevant interview segments. Analyses of the clinical interview were made, and copies of all relevant written work from the teaching experiment on pattern generalization were also included. I then developed and later synthesized individual cognitive maps with the aim of schematically capturing their generalizing schemes from problem to problem. Next, I compared, analyzed, and categorized the summative evidence using grounded theory that enabled me to make a case about the existence of the three labels corresponding to the three template types. I further engaged in several iterated processes of reading and analyzing the case studies in order to ensure the validity of the categorization of student work (written and oral) and obtain a sufficient characterization of each template type.

5 Results

Table 2 is a summary of the students' direct formulas on each pattern task and categorized by template type. Table 3 is a summary of the students' perceptions of what patterns meant to them. Each subsection below provides details of the students' responses on the X_3 tasks. The first subsection discusses briefly the 11 students' general perceptions of patterns in mathematics. The second subsection provides examples of simple additive-based pattern generalization templates that enabled the students to develop constructive nonstandard algebraic generalizations. The third subsection discusses the significance of multiplicative templates that further enabled them to produce constructive standard, constructive nonstandard, and deconstructive algebraic generalizations. In situations involving Figs. 2 and 3, in particular, their strategies reflect the use of pragmatic templates, which involves a combination of numerical and visual approaches. The fourth subsection provides examples of such instances in which case the algebraic generalizations they produced could be characterized under any of the above generalization forms discussed in Section 5.3.

5.1 Students' general perceptions of patterns in mathematics

From Table 3, five students articulated the necessity of having a direct formula in any pattern, while four others noted the increasing and decreasing relationship from stage to stage "in a certain way." The remaining two students noted the invariant figural structure across stages.

5.2 Additive templates and relevant mechanisms that lead to an algebraically useful structure

Additive constructive nonstandard generalizations Whole-number additive reasoning involves one level of abstraction, one inclusion relationship, and the repeated addition of ones (Fig. 6a). In the context of pattern generalization, an early additive visual template has a similar characterization and is evident when students count discrete objects one at a time without attending to any structure (cf. Rivera & Becker, 2008b, 2009a). A mature additive

Table 2 Summary of students' direct formulas categorized by template type (n=11)

Patterning task	Additive template	Multiplicative template	Pragmatic template
Square pattern in Fig. 1	Dung: $s = n + n - 1$	Dexter: $B = 2s - 1$	Anna: $n = 2s - 1$, $n = 2s - 2 + 1$
	Karen: $x = n - 1 + n - 1 + 1$	Diana: $n = 2x - 1$; $n = x^2 - (x - 1)^2$	Cherrie: $s = 2n - 1$; $x = 8n - 4$ (a second pattern)
	Emma: $s = n + n - 1$	Dung's 1st: $s = 2n - 1$	Frank: $S = 2x - 1$; $S = 2(x - 1) + 1$
		Emma: $s = n + 1$	
		Eric: $2(s - 1) + 1 = n$; $2s - 1 = n$	Shaina: $s = n \times 2 - 1$
		Karen: $x = 2n - 1$	Tamara: $x = 2p - 1$
Square frog pattern in Fig. 2		Anna: $g = 8s + 4 + s(s + 1)$	Karen: $x = n(n + 1) + [(n + 1) + n]4$
		Cherrie: $x = (n^2 + n) + 4(2n + 1)$	
		Dexter: $B = 4(2s + 1) + s(s + 1)$	Shaina: $s = g + 8$ (incorrect; counted just the four legs)
		Diana: $x(x + 1) + 4(2x + 1) = n$	
		Dung: $g = n^2 + n + 4n + 4n + 4$	
		Emma: $S = n(n + 1) + 4(2n + 1)$	
		Eric: $s(s + 1) + 4(2s + 1) = n$	
		Frank: $n(n + 1) + 4(2x + 1) = S$	
		Tamara: $x = n^2 + (n \times 8) + (4 + n)$	
Square array pattern in Fig. 3		Anna: $n = s(s + 1) + s(s + 1)$	Dexter: P=4 s (incorrect; just counted the perimeter sides)
		Cherrie: $x = n(n + 1)2$	
		Diana: $4x + 2(x - 1)x = s$	Emma: $s = n(2n + 2)$
		Dung: $s = [4n - (n - 1)n] - (n - 1)n$	Karen: $x = 4n + n(2n - 2)$
		Eric: $4s + [s(s - 1)]2 = n$	
		Frank: $s = x(x + 1)2$	Shaina: $s = n \times 8 - 4$ (incorrect)
		Tamara: $(p + 1)p + (p + 1)p = x$	

Table 3 Summary of the students' perception of patterns in general (n=11)

Verbal responses	Frequency
"The base stays the same and you add the stage number."	2
"You have the same base and you add the same amount every time."	
"There's a formula and it worked for every single one."	4
"If it increases or decreases in a certain way."	2
"Adds or subtracts and repeats the same amount."	
"If you break into parts, it's the same thing."	1
"Like the stage before it has the same shape."	1
"When it's hard to make a formula, it's not a pattern."	1

template, which is the main concern in this section, also has a similar characterization. However, what comprises a "one" is explained in the context of a *variable unit*. For example, Karen (seventh grader, cohort 2) extended the incomplete pattern in Fig. 1 to five stages by adding a square on each row and column per stage number, resulting in a growing L-shaped pattern (Fig. 9). In constructing and justifying her direct formula, $x = n - 1 + n - 1 + 1$, we obtain a glimpse of her additive template, as follows:

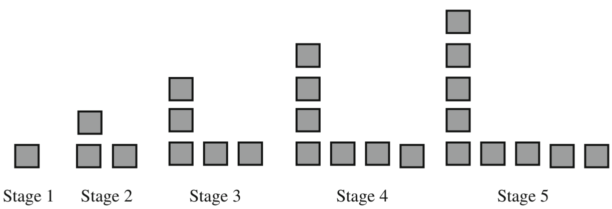
> Group these [*the row and column squares excluding the common corner square*] and add 1 [*the corner square*] (see ESM 1 for a visual illustration).

Karen's abductive–inductive action had her seeing each nonoverlapping set of squares as a variable unit that then allowed her to symbolically transition to a direct formula in additive form. In this situation, the cardinality (i.e., size) of each variable unit, $n - 1$, has been drawn from the manner in which Karen saw its relationship with the corresponding stage number. Further, while the size of her variable unit changed, its identity stayed the same from stage to stage.

Karen's additive template enabled her to produce and justify a constructive nonstandard generalization (CNG). The nonsimplified nature of the terms in a CNG signifies seeing parts that are related to each other relative to an interpreted structure. For example, in Karen's case, her interpreted structure of the growing L-shaped pattern in Fig. 9 involves seeing two equally sized nonoverlapping (horizontal and vertical) legs and a corner square in every stage, which explains why she had three terms in her direct formula.

Stage-determined generalizing process From the X_3 clinical interviews, only three of the total responses reflect the use of an additive CNG template in relation to the Fig. 1 task. Dung (eighth grader, cohort 1) and Emma (eighth grader, cohort 1) produced the same Fig. 9 pattern in relation to the Fig. 1 task, but their interpreted structure differed from

Fig. 9 L-shaped pattern in relation to the Fig. 1 task



Karen's abduction that led them to the additive formula $s = n + n - 1$. In the interview segment below, Emma demonstrates how she used her additive template in obtaining a CNG from the available stages (refer to ESM 2 for a visual illustration). Dung also provided the same reasoning as Emma.

FDR: What helped you in transitioning from these visual squares to a direct formula?

Emma: Grouping it, I guess. This is stage 1 [*referring to the one square*]. This is stage 2 [*the column of two squares*]. This is stage 3 [*the column of three squares*] and this is stage 4 [*the column of four squares*]. And then so ahm when I figured that, I try to see what's left. So if it's 1 [*the remaining square on the row of stage 2*], if you subtract the stage number from 1, you get 1. If you subtract 1 from the stage number [*stage 3*] you get 2 [*the two remaining squares on the row of stage 3*]. If you subtract 1 from this stage number [*stage 4*], you get 3 [*the three remaining squares on the row of stage 4*].

In the case of Emma and Dung, each figural stage consists of the union of two variable units having cardinalities n and $(n - 1)$ corresponding to the column and row of squares, respectively. In their situation, while the two units are nonidentical, they are still related in a particular way.

Pattern goodness All 11 interviewees produced the same L-shaped pattern shown in Fig. 9 in relation to the Fig. 1 task. Also, they were successful in coordinating their actions relative to the Fig. 9 pattern, producing CNG and several more other types, which we discuss in the next section. Suffice it to say, Fig. 9 represents a pattern that is *high in Gestalt goodness* in that it consists of stages whose basic structure has cohesiveness (i.e., in the sense of belonging together) and clearly redundant parts (i.e., repetition of the same part), which also means to say that the pattern has a structure that enabled the students to "specify a geometric or algebraic formula that st[ood] out through its simplicity" (Metzger, 2006, p. 24).

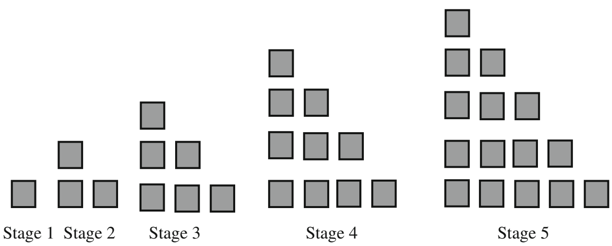
While an additive template is helpful in constructing and justifying a CNG, it is considerably difficult to apply in figural patterns that are *low in Gestalt goodness*. For example, Emma's third extension relative to the Fig. 1 task is the triangular pattern shown in Fig. 10. In the interview segment below, she tried to use an additive template in constructing a direct formula.

Emma: I know what to write but I don't know how to put it into a formula. Like so if there's n and then you add all the numbers before it. Like if it's 5, you put $n - 1$, $n - 2$, $n - 3$, and $n - 4$. But I don't know how to put that in a formula.

FDR: So if it's stage 6, if you use your formula, it is?

Emma: It's n plus $n - 5$ plus $n - 4$ all the way to 1.

Fig. 10 Emma's third pattern in relation to the Fig. 1 task



For Emma, her interpretive structure of Fig. 10 necessitates further conceptualization. Her situation exemplifies the mutually determining roles of pattern goodness and visual template use in pattern generalization, a finding that is further illustrated in the next two sections below.

5.3 Multiplicative templates and relevant mechanisms that lead to an algebraically useful structure

Constructive standard generalizations Whole-number multiplicative reasoning involves simultaneously coordinating at least two levels of abstractions and inclusion relationships (see Fig. 6b). In the context of pattern generalization, such coordination is articulated when students see and count multiples (or copies) of the same variable unit(s). For example, Diana (seventh grader, cohort 2) saw each of the four frog legs in the Fig. 2 pattern as the union of a corner square and two copies of the same variable unit that enabled her to claim the direct expression $2x+1$: "[*Referring to stage 3*] 3, 3, and 1, so 2x+1." This particular algebraic generalization is classified as a *constructive standard generalization* (CSG). (Refer to ESM 3 for additional examples of CSG.)

Deconstructive and auxiliary-based constructive nonstandard generalizations Diana's thinking relative to the Fig. 1 task also exemplifies the use of a multiplicative template. She produced the same two patterns shown in Figs. 9 and 10. In the case of Fig. 9, Diana obtained two direct formulas that initially showed her inclination toward a multiplicative template. In the interview segment below, she described how she thought about her two formulas.

FDR: Is there a formula here [*referring to the Fig. 9 pattern*]?

Diana: Uhm, $n = 2x - 1$. This is step, stage 3. There's like 2 threes and this one [*corner square*] overlaps. [Refer to ESM 4 for a visual illustration.]

FDR: So for stage 4? [*Diana gestures two groups of four squares from the figure and the corner square*.] Is there another formula for that?

Diana: Uhm, I'm not sure this is the simplest form. $n = x^2 - (x - 1)^2$. Like if you see a square and a square here. [*Refer to stage 4 in Fig. 11. Diana made a gesture to indicate that she was taking away* $(4 - 1)^2$ *or 9 squares*].

Diana's multiplicative template had her initially seeking out variable units, connecting them with the appropriate stage number, and finally expressing their totality in multiplicative form. Her direct formula relative to Fig. 9, $n = 2x - 1$, exemplifies a *deconstructive generalization* (DG), that is, she interpreted the general L-shaped structure

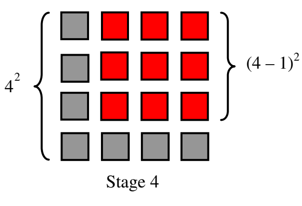


Fig. 11 Diana's visual demonstration of the formula $n = x^2 - (x - 1)^2$ in relation to the Fig. 10 pattern

of her pattern in terms of two overlapping legs with a corner square that she counted twice, which explains why she had to take away one square (i.e., the constant term −1). Karen's second direct formula, $s = 2n - 1$, in relation to the Fig. 9 pattern was also a DG. In justifying her DG, she clearly articulates her multiplicative template, as follows (see ESM 4):

> I visualized it in groups. So like for the 2n – 1, you take the stage number which are these two [*the two circled groups in every stage*] and then you subtract 1 because there's an overlap.

The written work of Tamara (eighth grader, cohort 1) and Selma (eighth grader, cohort 2) in Fig. 7 also demonstrates the use of a DG.

Diana's other formula $n = x^2 - (x - 1)^2$ exemplifies the use of an auxiliary set on a route to establishing a multiplicative-based CNG. As illustrated in Fig. 11, she initially added an auxiliary set of 3^2 squares to complete the larger square (i.e., 4^2), which she then took away.

In the case of Fig. 10, the direct formula that Diana constructed occurred in joint activity with me due to the fact that she was unable to successfully implement her multiplicative template in a nonlinear pattern that to her was low in Gestalt goodness. Initially, I had her investigating a relationship between a particular figural stage and the complete rectangle formed by joining two copies of the same stage (see Fig. 12 for an illustration involving stages 2 and 3 of the pattern). We used the square blocks on the table in accomplishing the abductive–inductive action that enabled her to infer a relationship and transition to symbolic action. She then concluded that an appropriate direct formula is $n = \frac{x(x+1)}{2}$. This particular generalization strategy that I used with Diana in relation to the Fig. 9 pattern also exemplifies the use of an auxiliary set in the construction and justification of an algebraic generalization.

Cherrie (eighth grader, cohort 1) employed a multiplicative template in relation to the Fig. 3 task and produced a CNG. (Refer to ESM 5 for her written work.) She initially interpreted each stage in the pattern in terms of multiples of horizontal rows and vertical columns of sticks. She then counted in the following manner: "[*In stage 2*] there's 2, 2, 2, 2, 2, and 2; [*in stage 3*] there's 3, 3, 3, 3, 3, 3,3, and 3; so there's two groups of 3 twice.... There's two 4 groups of 3 [*referring to stage 3*], two five groups of 4 [*referring to stage 4*]... so x equals two times n+1 times n." She then wrote, "$x = n(n + 1)2$."

Figural parsing The strength of Diana's multiplicative template became especially evident in the case of the Fig. 3 pattern. Initially, she obtained a formula for the number of sticks on the perimeter of each square, $4x$. Next, she counted the interior sticks as follows: (1) "[*in stage 2*] 2 minus 1 would be 1 so there would be 1 row going down and another row so that would be rows of 2 sticks;" (2) "in stage 3, there's 2 rows of 3 sticks;" (3) "[*in stage 4*

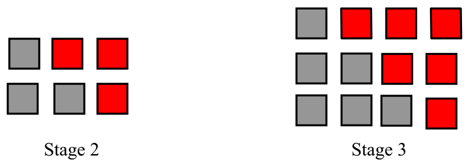


Fig. 12 Rectangles associated with stages 2 and 3 of the pattern in Fig. 10

there's] three, okay, it's two sets of three rows of 4." She then wrote $4x + 2(x - 1)x = s$, a CNG, which she then simplified to $2x + 2x^2 = s$.

Like Diana, Dung also employed a multiplicative template in his pattern generalization of the Fig. 3 task. Unlike Diana who developed a CNG, Dung approached the task using a DG. Figure 13 provides a visual illustration of how he figurally parsed a stage into parts of separate rows of squares and separate smaller squares per row. He used stage 4 to aid him in performing a general abductive action, that is, he first parsed the figure into four disjoint rows and counted the number of sticks per row. In counting the number of sticks per row, he saw four disjoint squares for a total of $4 \cdot 4 = 16$ sticks and subtracted the three overlapping vertical sticks. He counted the total number of horizontal and vertical sticks counting repetitions and obtained $(4 \cdot 4 - 3) \cdot 4 = 52$. In his written work, he immediately resorted to the use of a variable n to convey that he was thinking in general terms, hence, the expression $(4n-(n-1))n$. Next, since the four disjoint rows had overlapping sides (i.e., the interior horizontal sticks), he took away three (= 4 – 1) groups of such four horizontal sticks from 52. This concrete instance allowed him to complete his DG, that is, $s = (4n - (n - 1))n - (n - 1)n$, which he then simplified to $s = 2n^2 + 2n$.

Sequence of multiplicative templates Dung's pattern generalization process in Fig. 13 could also be interpreted as having employed a sequence of DGs before arriving at a final algebraic generalization. Diana also used a sequence of actions. In the interview segment below, she employed a multiplicative template in the case of Fig. 2. While her final direct

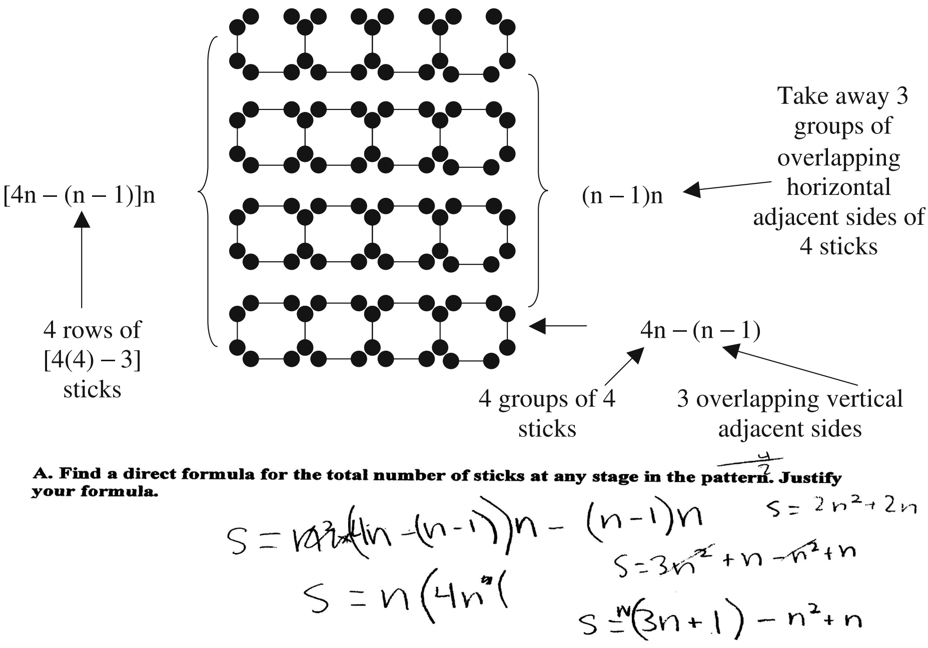


Fig. 13 Dung's construction and justification of his formula for the Fig. 3 pattern

formula took the form of a CSG, her thinking reflects a combined use of CSG, DG, and CNG, as follows.

Sequence of abductive–inductive actions	Sequence of symbolic actions
Diana: Well, basically you always, like, to this number here, to this part here [*referring to stage 2 in Fig.* 2], you added 1 and on this side you add 1 to make it longer [*referring to the growing legs on every corner.*] You always add 1 to everything to make the legs longer. Instead of like 2×2, you make it 3×3. And for this one, too [*the middle rectangle*], instead of 1 by 2, you make it 2×3. [*She then finds a direct formula and obtains* $x(x+1)+(2x+1)=n$].	
FDR: Okay, so tell me what's happening there? Where did this come from, $x\ (x+1)$?	CNG: $x\ (x+1)$
Diana: This, the little square, x times $x+1$.	
FDR: So where's the x times $(x+1)$ here [*referring to stage 3 in Fig.* 2]?	
Diana: Like 3×3, or 3×4.	
FDR: So where's the $2x+1$ coming from?	
Diana: This. I mean I can look at it like 2 times $(x+1)$ minus 1 but I just made it, like, 3, 3, and 1, so $2x+1$. [*She initially saw that each leg had two overlapping sides that shared a common corner square.*]	DG: $2(x+1) - 1$ CSG: $2x+1$
FDR: But this $2x+1$ is just for this side [*referring to one leg*], right?	
Diana: For all of the legs, oh, [*then adds a coefficient of 4 to her formula*: $x(x+1)+4(2x+1)=n$]	CNG : $x(x+1)+4(2x+1)=n$
FDR: Okay, so are you happy with your formula?	
Diana: I think I could simplify it. I'd like to see what happens if I simplify it. [*She then simplifies her formula to* $x^2+9x+4=n$.]. 4 would be these [*the corner middle squares*] I'm pretty sure. $9x$ would be, 8, oh, yes, I see it. I see how it works. There's an x squared here [*referring to the rectangle which she saw as the union of an n by n square and a column side of length n*] if you see one square here and the $9x$ would be these legs [*referring to the (n+1)th column of the rectangle of length n and the eight row and column legs minus the corner middle squares*]. Plus 4 would be the center of each leg.	CSG: $x^2+9x+4=n$

5.4 Pragmatic templates and consequences

Numerically driven pragmatic templates and consequences Pragmatic reasoning involves flexibly or harmonically employing a combination of different approaches such as visual and numerical strategies in pattern generalization. Emma's thinking in relation to the Fig. 3 task provides a good starting illustration of pragmatic templates. (Refer to ESM 6 for her written work.) She began inspecting parts in a figural stage that mapped with the corresponding stage number and then tried to use an additive template. When that did not work for her, she then claimed that since "stage 1 had 4 sticks... it is like four of stage 1." In stage 2, she counted 12 sticks in total and then abducted the following claim: "[*stage 2*] • 6." She then performed induction by using stages 3 and 4, that is: since stage 3 has 24 sticks, then it is equal to "[*stage 3*] • 8;" and since stage 4 has 40 sticks, it could be seen as "[*stage 4*] • 10." In transitioning to symbolic action, she inferred the direct formula

$s = n \cdot (2n + 2)$. In justifying her formula, Emma claimed that the variable n referred to stage number and that the numbers 4, 6, 8, and 10 were "even numbers and that to get to 10, I multiply it by 2 and add 2, so $2n+2$." But when I asked her to explain how her direct formula might make sense in the given figural stages, she said, "I don't know, I don't see it in the picture." Emma's pragmatic template in this case was a combination of numerical and visual approaches. However, the dominance of the former approach consequently prevented her in justifying her formula.

Karen also used a numerical approach when she was unable to make sense of the Fig. 3 pattern visually. (Refer to ESM 7 for her written work.) Initially, she saw that the total number of sticks on the perimeter of each square was $4n$ ("there are four sides... four groups of the stage number"). She then noticed that the interior sticks had the following relationship, which she organized in a table: stage 1 has 0 sets of horizontal and vertical sticks; stage 2 has 2 sets of 2 sticks; stage 3 has 4 sets of 3 sticks; stage 4 has 6 sets of 4 sticks;...; and stage n has $(2n-2)$ sticks. In obtaining the expression $2n-2$, she employed a numerical strategy. Unfortunately, the dominance of the numerical approach, as in Emma's case above, prevented her in justifying her direct formula, $x = 4n + n(2n - 2)$.

Some of those students who employed a pragmatic template, in fact, relied more on numerical than visual approaches. For example, Anna's (eighth grader, cohort 1) pragmatic template below allowed her to correctly justify figural patterns that to her were high in Gestalt goodness such as the one shown in Fig. 9.

> Like if I look at it visually, then, or like if I write it out in numbers like how much is added on, then I use that amount or number. Then I try to like use the stage number, too, since it's like it's kinda part of the equation. And then ahm since you usually multiply it and you add or subtract the amount that you need to get the stage amount.

However, she found it difficult to employ her template above in the case of Figs. 2 and 3. As shown in Fig. 14, Anna initially tried a differencing-table strategy that she found difficult to use in constructing a direct formula using her template above. Unlike Frank below, however, she decided to use a visual strategy that conveys the use of a multiplicative template.

Frank (eighth grader, cohort 1) consistently employed a pragmatic template in Figs. 1, 2, and 3. He explained his template in the following way:

> I'd looked through the differences of each pattern, say [*stages*] 1 and 2 [*referring to the pattern in Fig.* 9]. You notice that there's two more than before. So when I look at that, I wanna multiply it by the number that's different because you want to get to the next stage, right? And that's how many is needed to get to the next stage of your pattern.

However, because the numerical approach was more dominant than the visual in his pragmatic template, his justification focused on how each factor and term in his direct formulas actually assisted him in constructing his figural stages in a step-by-step manner without attending to structural issues (invariance, unit, etc.). For example, Frank's extension of the Fig. 1 task is shown in Fig. 9. He obtained his two direct formulas $S = 2x - 1$ and $S = 2(x - 1) + 1$ as a result of seeing a common difference of two squares ("there's two more than before") from one stage to the next. The two interview segments below provide his justifications.

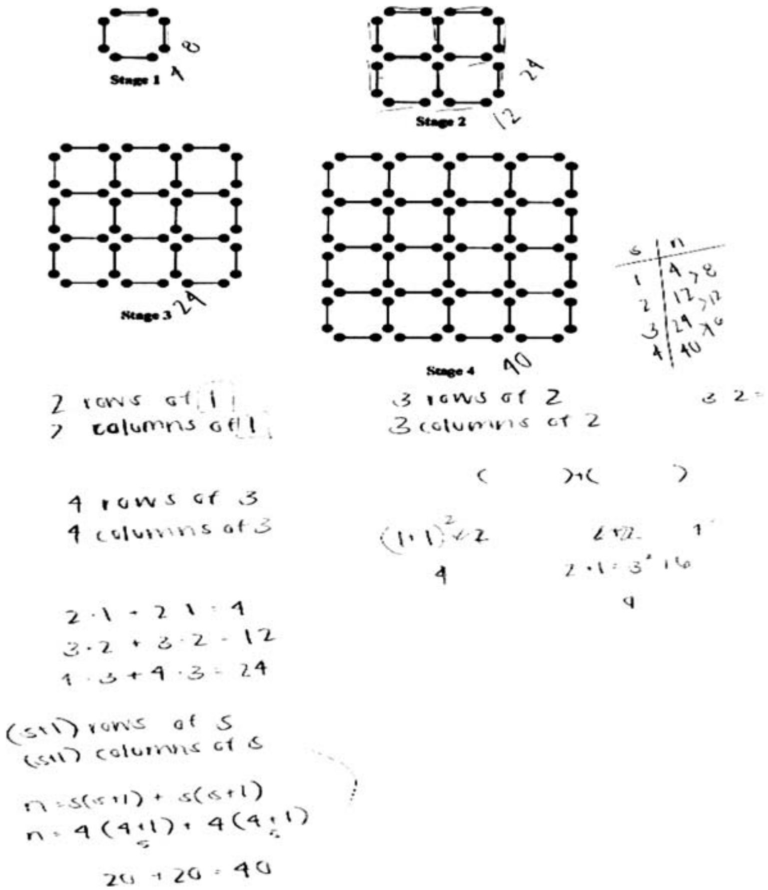


Fig. 14 Anna's written work in relation to the Fig. 3 pattern task

Frank's justification of the formula $S = 2x - 1$ (see ESM 8 for a visual illustration)

Frank: In stage 2, 3 total, so if I had another 1, that would equal 4 so subtract 1 to get to stage 3.

FDR: What about for stage 3?

Frank: You would have ahm so 2 times 3 is 6 there would be an extra so you could put wherever. [*He puts the extra square at the end of the column*.] But then you subtract 1 which is the extra to get to 5.

FDR: Okay, so one more.

Frank: So for stage 4, it has a total of 7. And then there's 1 more [*puts the extra square at the end of the column*] so you subtract that extra to get your total.

Frank's justification of the formula $S = 2(x - 1) + 1$ (see ESM 9 for a visual illustration)

FDR: Show me how you make sense of that formula here.

Frank: So if it's 2 minus 1, that gives you 1, multiply by 2 is 2 plus 1 gives you 3.

FDR: So where is that there [*referring to the stages*]?

Frank: So say, stage 2, right [*starts with 2 squares*]? And then 2 minus 1 is 1 [*takes away 1 square*] and then you multiply 1 by 2 which gives you that [*puts another square*] and then you add 1 there.

FDR: Hmm, so let's try that for this one [*stage 3*].

Frank: So it would be 3 [*starts with 3 squares*] for this one. And then you subtract 1 which gives you a total of two squares. 2 times 2 gives you 4 and then you're missing one so you add 1 more.

Tamara's (eighth grader, cohort 1) extended pattern in Fig. 16 relative to the Fig. 1 task exemplifies a situation in which the use of a pragmatic template did not yield an algebraic generalization. Figure 15 shows portions of her written work relative to the Figs. 2 and 3 tasks that reflect consistency in the use of a multiplicative template leading to two CNGs resulting from a figural parsing strategy. In the case of Fig. 16, she first compared stages 1 and 2 and saw that stage 2 had "one group of two squares" added to stage 1. She then pursued the following structure: stage 3 would have "two groups of two squares" added to stage 2; stage 4 would have "three groups of two squares" added to stage 3; stage 5 would have "four groups of two squares" added to stage 4; etc. In thinking about a direct formula, she first tried to visually find a way of linking the extra square and the two equal rows of squares that she inferred on her pattern. When asked to explain her thinking, she claimed that she was trying to "find a way to relate grouping and stage number in some way." When her visual attempts failed, she switched to a numerical approach and produced the following sequence: ($2 \cdot 1-1$, $2 \cdot 2-1$, $2 \cdot 4-1$, $2 \cdot 7-1$, $2 \cdot 11-1$, $2 \cdot 16-1$). She then set up a table of values and tried to make sense of the subsequence (1, 2, 4, 7, 11,...) by relating each term with the associated stage number. Feeling frustrated, she abandoned her Fig. 16 pattern, produced the pattern shown in Fig. 9, and then obtained the formula $x = 2p - 1$, a DG.

Pragmatic templates that demonstrate an effective synergy between numerical and visual strategies There were students who also employed a pragmatic template but were successful in maintaining a balance between numerical and visual approaches. For example, when Cherrie was presented with the Fig. 1 task, she produced the same pattern shown in Fig. 9. She then established her direct formula, $x = 2n - 1$, numerically by noting that the pattern "always adds by 2." In her justification below, her reasoning conveys the

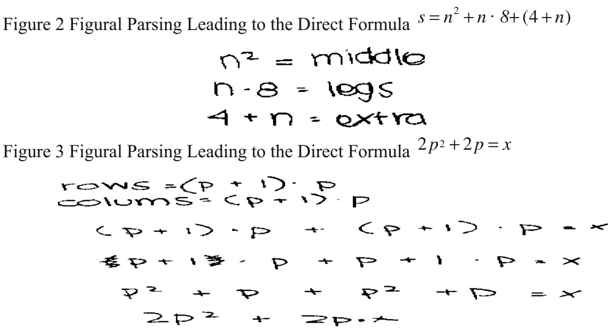


Fig. 15 Tamara's pattern generalization in relation to the Figs. 2 and 3 tasks. Figure 2 figural parsing leading to the direct formula $s = n^2 + n \cdot 8 + (4 + n)$. Figure 3 figural parsing leading to the direct formula $2p^2 + 2p = x$

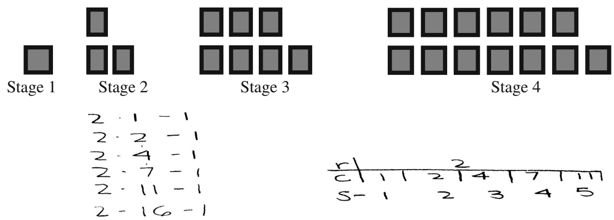


Fig. 16 Tamara's extension of the stages in Fig. 1

use of an auxiliary set that enabled her to explain why and how her multiplicative-based CSG made sense to her (*see ESM 10 for a visual illustration*).

> Cherrie: There's a pattern, so it always adds by 2. So, hmm, oh okay got it. OK. Uhm, $x = 2n - 1$.
>
> FDR: So what do these variables mean, the x and n?
>
> Cherrie: x is the amount of blocks and n represents the stage number.
>
> FDR: Okay. So how do you justify your formula?
>
> Cherrie: So $2n$, because it increases by 2, so you times it by 2 and then you minus 1 [*i.e., a row and a column having the same number of squares and then taking away 1 block*]. It's like a missing block like the one in the middle [*points to the missing block in stage 2*], there [*points to the missing block in stage 3 after making a gesture that conveyed rotating the row of squares into a column of squares*], there [*same pointing-gesturing act*].
>
> FDR: You started with those set of stages in your pattern. And then you're able to successfully come up with a formula in symbolic form by using algebra. So what helped you?
>
> Cherrie: Well... since you came in and then you introduced us to formulas, and then you introduced us to patterns. Like how, like 1 to 3, but how is it increasing? Then 3 to 5, how is it increasing? And then we... it says by 2. And then so we say if it's increasing, then you times it by what, the stage number. So if it's like 1 [*referring to stage 1*], and then it's 2 and 3 [*counting on two more blocks in stage 2*], right? So you go two times your stage number, which is 1 [*referring to stage 1*], and it's 2. But because it's [*the result*] like a block more, so you have to minus 1 to equal the figure [*the 1 block in stage 1*].

Pragmatic templates that combine additive and multiplicative templates Karen's pattern generalization of the Fig. 2 task, $x = n(n+1) + [(n+1) + n]4$, exhibits a pragmatic template that combines additive and multiplicative templates. She initially saw a middle rectangle with dimensions n and $(n+1)$. She then interpreted each of the four L-shaped legs to be the union of a column side with $(n+1)$ squares and a row side with n squares.

6 Discussion

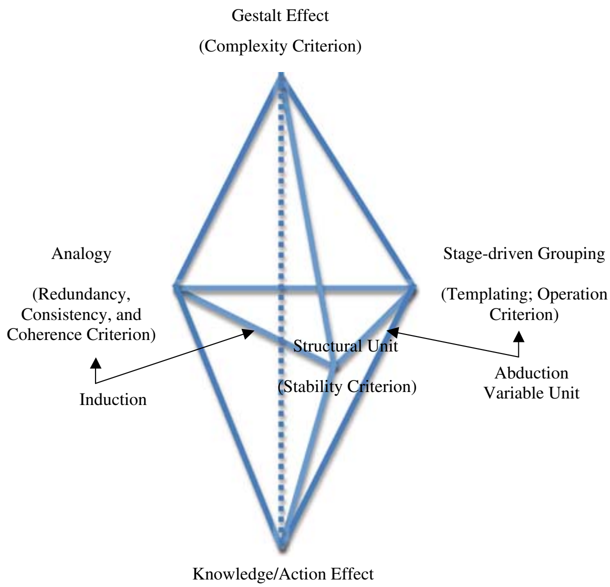
Revisiting students' general perceptions of patterns A tough question, but one that cuts through much of constructivist thinking involving pattern generalization, is the following:

How is it possible for learners to obtain, and even more so justify, an algebraically useful generalization when some relevant structure is in the process of being determined? In other words, which comes first to learners: structure or formula? Arnheim (1971), who addresses similar abstraction issues, claims that "it requires a mind that, in perceiving a thing, is not limited to the view it receives at a given moment but is able to see the momentary as an integral part of a larger whole, which unfolds in a sequence" (p. 50). On the basis of findings drawn from this study, an operational, empirical response involves visual templates that operate like mental strips (in Hutchins's sense) that the 11 students demonstrated and articulated within and across task. Visual templates exist and individually could be seen as a product of a distributed and coordinated phenomenon involving several factors such as the interaction between abductive–inductive and symbolic actions (Fig. 5) or, more generally, those identified in Fig. 17, which we discuss in the next subsection below.

The 11 students' canonical images of patterns, which they described in words in Table 3, provide a sufficient initial context that enables us to make sense of their visual templates. Further, their templates could be characterized as being primarily stage-determined ("visualizing in groups of the stage number;" "grouping by the stage number;" "x rows of the stage number;" "A groups of the stage number;" etc.) that they clearly verbalized during the X_3 clinical interviews. Following Neisser (1976) and as demonstrated consistently in the preceding section, the students' canonical images relevant to pattern generalization provided them with a structure or model that they used in processes of comparing, contrasting, and extending. Overall, they knew that individual patterns consist of stages that (1) increase or decrease in a particular way and (2) share similar parts or redundant parts (i.e., variable units) that (3) could then be symbolically captured in some closed formula.

A primary issue with such a canonical image of patterns is the relative ease in which (1) and (2) could be accomplished by way of construction and creative imagery but the relative

Fig. 17 A 3D cognitive model of a visual template involving pattern generalization



difficulty in fulfilling (3) due to pattern goodness. Indeed, patterns that are low in Gestalt goodness require more conceptualization in order to be representationally adequate, which means to say that an interpreted algebraically useful structure would necessitate more work. The 11 students were all well aware of the need to produce stages in their patterns that had parts that were intrinsically good and algebraically useful. Tamara's failed pattern generalization relevant to Fig. 16 exemplifies a situation of discord among (1), (2), and (3). Certainly, her swift decision to shift to a figural pattern that was high in Gestalt goodness was made possible by her canonical image of a pattern that had to satisfy all three elements of (1), (2), and (3).

Revisiting the nature of visual templates Tamara's situation in Fig. 16 also underscores the incomplete, skeletal, and constructionist nature of visual templates. While such templates provided my students with useful mental strips and purposeful visual-mode-rated sequences of actions, I note both the influence of either the content of their inferred assumptions or prior mathematical knowledge and the necessity of acquiring more pattern-generalizing strategies that matter in pattern-related actions of focusing (Lobato, Ellis, & Muñoz, 2003), specializing (Mason, Graham, & Johnston-Wilder, 2005), and figural parsing (Friel & Markworth, 2009; Rivera & Becker, 2009a; Rivera, 2007, 2009). For example, the three students (Diana, Emma, and Dung) who produced the triangular pattern in Fig. 9 and Tamara in the case of her Fig. 16 pattern assumed that any constructed pattern of their making could be easily captured in symbolic form by simply employing either a differencing strategy (Orton & Orton, 1999, p. 107) or the use of an inductive table (Rivera, 2009), which actually becomes very complicated to use in the case of nonlinear patterns whose direct formulas are polynomials in form.[6]

Drawing on my experiences with (1) Dung, Emma, and Diana in relation to the Fig. 10 pattern, (2) Tamara and her Fig. 16 pattern, and (3) Cherrie's pragmatic generalization and visual justification of the Fig. 9 pattern, students need to know when, how, and why it is necessary to introduce auxiliary sets or, more generally, employ *figural transformations* or what Duval (2006) refers to as the action of *figural change* in apprehending a (geometric) figure. Visual actions of moving, reorganizing, and transforming (Liang Chua & Hoyles, 2009; Rivera, 2007, p. 72), including the use of auxiliary sets and figural parsing in this article, are powerful mathematical ways of inferring a plausible structure on a pattern. (ESM 11 shows one possible figural transformation of Tamara's pattern in Fig. 16 that produces an algebraic generalization.)

Revisiting the Fig. 5 diagram: a 3D visual template model in pattern generalization activity Figure 17 is a three-dimensional cognitive model of the visual template that I have drawn from the clinical interviews with the 11 students. It is a refinement of the model shown in Fig. 5 that I inferred from clinical interviews in two earlier studies I conducted with preservice elementary majors (Rivera & Becker, 2007) and the cohort 1 students over two consecutive years of constructivist-driven design experiments on pattern generalization (Rivera, 2008). Considering the above findings, a more efficient and coordinated visual template in pattern generalization activity has the following five properties that operate in a distributed and dynamically embedded manner: structural unit, analogy, stage-driven grouping, pattern goodness, and knowledge/action effects.

Gestalt effect addresses issues relevant to pattern goodness; it is a basic structural criterion for algebraic usefulness. *Knowledge/action effect* is related to prior and

[6] *ESM* 12 shows two numerical approaches in dealing with Tamara's Fig. 16 pattern.

available mathematical knowledge (e.g., differencing; the use of inductive-structuring tables) and figural-related actions (e.g., focusing; specializing; transforming figural stages) that influence and are influenced by pattern goodness perception. Following Trudeau and Dixon (2007), such actions "create meaning" and "form relational representations."

The dashed line between the two effects symbolically conveys possible shifts in template processing. For example, the pattern generalizations of Karen and Emma exemplify a shift from the visual (say, their thinking about Fig. 9) to the pragmatic and then to numerical (say, their thinking relative to Figs. 2 and 3 tasks). The pattern generalizations of Frank (say, his work on Fig. 9) and Anna (say, the beginning part of her written work in Fig. 14) illustrate a route that begins at the opposite end of the spectrum (i.e., using numerical knowledge).

When pattern goodness is high as in Fig. 9, there is little to no need for either additional mathematical knowledge or a more elaborate figural action since the stages have a shared immediate property that could be referred to as a *visual pop-out*, that is, it has an interpreted structure that could be done rapidly without effortful scrutiny (cf. Leslie, Xu, Tremoulet, & Scholl, 1998, p. 11). However, when pattern goodness is low, then additional content knowledge (e.g., using differencing and inductive-structuring tables shown in ESM 12) and/or figural action (e.g., Dung's complex parsing in Fig. 13) would be needed in order to make an interpreted algebraically useful structure of a pattern become more familiar to the perceiver. Metzger (2006) notes this as well when he claims that knowledge, experience, and behavior become "even more decisive factors the less simple the structure of the image that requires completion" (pp. 136–137). Dung and Diana effectively parsed visually, which explains their consistent use of visual templates throughout all three tasks. Emma and Karen in relation to the Fig. 3 task resorted to using their knowledge of numerical differencing in constructing their CNGs despite their lack of knowledge about a plausible structure of the pattern. Their failure to justify could be accounted for by their relative inability to effectively parse the given figural stages.

Structural unit pertains to a perceived interpreted general identity of a pattern within and across stages. It gives meaning to the variable units that are constructed and addresses issues of invariance and change. Thus, it addresses the issue of stability of form or shape, property, attribute, or relationships in a pattern. *Analogy* addresses redundancy, consistency, and coherence among parts in a pattern. *Stage-driven grouping* shapes the content of an emerging template and, consequently, addresses operations that are used reflective of the (sequence of) grouping actions that are performed within and across stages. Central to the triad are the abductive–inductive and symbolic actions that require effective coordination as well.

The 11 students' success in pattern generalization (Table 2) involving simple and complex figural patterns underscores the intricate interconnectedness of the three properties (or criteria) above, that is, they mutually determine each other. While inspecting for groups of the stage number in a figural stage is a useful action in developing a structural unit, performing repetitions of the same action in several more stages assist in establishing analogical relatedness across stages. Further, what constitutes a structural unit is, of course, an individual experience, which explains the existence of (at least seven) different forms of algebraic generalization. For example, the students produced at least three different structural units relative to the Fig. 9 pattern, and they expressed them symbolically in different (but equivalent) ways using either an additive or a multiplicative template. Finally, Gestalt effect introduces a complexity in the implementation of the triad with those low in pattern goodness requiring more knowledge and/or figural action.

Revisiting additive, multiplicative, and pragmatic templates While the use of an additive template in this study produced CNGs, a multiplicative template spawned CNGs, CSGs, and DGs. Furthermore, sequences of relevant template-driven actions (e.g., the sequence of CNG, DG, and CSG in the case of Diana relative to the Fig. 2 task; the sequence of DGs in the case of Dung in Fig. 13) were evident in cases of figural patterns with an inferred complex structure. I also underscore the inadequacy of an additive template in most cases of figural patterns that have (subjectively) more complex structures (e.g., Emma's predicament in relation to Fig. 10) and the representational power of a multiplicative template in dealing with (almost) all linear and a few quadratic types of pattern generalization tasks. Finally, the genesis of pragmatic template use could be traced to either one of the following two situations, which has been taken into account by the vertical components in the Fig. 17 model: (1) when some students were unable to effectively employ increasingly complex figural strategies (parsing, transformations, etc.) in order to cope with a pattern that they found to be low in Gestalt goodness, it then signaled them to explore and shift to a numerical strategy; or (2) when they began the pattern generalization process using a numerical strategy (e.g., differencing) and then justified their constructed direct formulas visually through induction. One common observation among those students who employed a pragmatic template involves the fragile nature of their justifications. For example, the justifications that were offered by Cherrie (ESM 11) and Frank above in relation to the Fig. 9 pattern exemplify a type of inductive reasoning strategy that could be valid in some cases and invalid otherwise. In most cases of pragmatic template use, the students performed constructing and justifying an algebraic generalization as two separate activities with a numerically driven pragmatic template as oftentimes skipping abductive action, which is central to structure discernment and formation.

7 Conclusion

In this article, I provided empirical evidence that sufficiently illustrates the existence of additive, multiplicative, and pragmatic templates in pattern generalization activity involving figural patterns among a group of middle-school learners. However, I underscore the complicit role of visualization and multiplicative reasoning in developing one systematic approach to pattern generalization. Perhaps arguable, while prevailing research studies in patterning activity conducted with younger and older students and adults (teachers) have articulated in varying degrees the importance of visualization in constructing direct formulas, the implications of multiplicative reasoning—especially the concept of a unit that is a unifying concept in algebra—have not been addressed and explored explicitly as one possible learning trajectory toward the development of students' pattern generalization skills. The complexity of coordinated actions that are involved in such trajectory can be dealt with by addressing the mutually determining (i.e., distributed) relationships between and among the various components of the visual template model in Fig. 17.

Four issues are worth pursuing further. *First*, while pattern goodness is not a new finding, how to assist students to cope with it (e.g., how to perceptually and symbolically infer an algebraically useful structure) is relatively new. More (large-scale) empirical investigations on the use of visual strategies and actions like figural transformations (figural parsing, introduction of auxiliary sets, etc.) and multiplicative reasoning in pattern generalization are needed to determine their overall effectiveness in resolving this issue. Again, I emphasize the crucial role of multiplicative (more than additive) schemes in pattern generalization with its focus on the concept of a unit that is central in pattern structure, formation, and discernment. *Second*, further teaching experiments are needed in order to generate more visual strategies involving figural transformations beyond those already discussed in this article. *Third*, while this article demonstrates the existence and effectiveness of visual templates in dealing with figural patterns that have linear and relatively simple quadratic structures, further research is needed in ascertaining the possibility of visual templates in all figural patterns that have a nonlinear structure. *Fourth*, it is interesting to explore consequences of the Fig. 17 model relative to task construction and, more generally, the teaching and learning of algebra. For example, how can the model be used to help insert tasks involving pattern generalization into the teaching and learning of, say, problem solving that target functions and mathematical modeling?

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215–241.

Arnheim, R. (1971). *Visual thinking*. Berkeley: University of California Press.

Becker, J. R., & Rivera, F. (2005). Generalization schemes in algebra of beginning high school students. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (pp. 121–128), vol. 4. Melbourne, Australia: University of Melbourne.

Berkeley, I. (2008). What the <0.70, 1.17, 0.99, 1.07> is a symbol? *Minds and Machines, 18*, 93–105.

Billings, E., Tiedt, T., & Slater, L. (2007). Algebraic thinking and pictorial growth patterns. *Teaching Children Mathematics, 14*(5), 302–308.

Bourdieu, P. (1999). Understanding. In P. Bourdieu (Ed.), *The weight of the world: Social suffering in contemporary society* (pp. 607–626). Cambridge: Polity Press.

Clark, F. & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1–5. *Journal for Research in Mathematics Education, 27*(1), 41–51.

Davis, R. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. New Jersey: Ablex.

Davis, R. (1993). Visual theorems. *Educational Studies in Mathematics, 24*, 333–344.

Dörfler, W. (2008). En route from patterns to algebra: Comments and reflections. *ZDM, 40*(1), 143–160.

Dretske, F. (1990). Seeing, believing, and knowing. In D. Osherson, S. M. Kosslyn & J. Hollerback (Eds.), *Visual cognition and action: An invitation to cognitive science* (pp. 129–148). Cambridge: MIT Press.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*, 139–162.

Friel, S. & Markworth, K. (2009). A framework for analyzing geometric pattern tasks. *Mathematics Teaching in the Middle School, 15*(1), 24–33.

Gattis, M. & Holyoak, K. (1996). Mapping conceptual relations in visual reasoning. *Journal of Experimental Psychology. Learning, Memory, and Cognition, 22*(1), 231–239.

Giaquinto, M. (2007). *Visual thinking in mathematics*. Oxford: Oxford University Press.

Healy, L. & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers? *Mathematical Thinking and Learning, 1*(1), 59–84.

Hutchins, E. (1995). *Cognition in the wild*. Cambridge: The MIT Press.

Koedinger, K. & Anderson, J. (1995). Abstract planning and perceptual chunks: elements of expertise in geometry. In B. Chandrasekaran, J. Glasgow & N. Hari Narayanan (Eds.), *Diagrammatic reasoning: Cognitive and computational perspectives* (pp. 577–626). Cambridge: The MIT Press.

Krutetskii, V. (1976). *The psychology of mathematical abilities in school children*. Chicago: University of Chicago Press.

Lannin, J. (2005). Generalization and justification: the challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7*(3), 231–258.

Lannin, J., Barker, D., & Townsend, B. (2006). Recursive and explicit rules: how can we build student algebraic understanding. *Journal of Mathematical Behavior, 25*, 299–317.

Lee, L. (1996). An initiation into algebra culture through generalization activities. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 87–106). Dordrecht: Kluwer.

Leslie, A., Xu, F., Tremoulet, P., & Scholl, B. (1998). Indexing and the object concept: Developing "what" and "where" systems. *Trends in Cognitive Science, 2*(1), 10–18.
Liang Chua, B., & Hoyles, C. (2009). Generalization and perceptual agility: how did teachers fare in a quadratic generalizing problem? In M. Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics* (pp. 13–18), vol. *29*(2).
Lobato, J., Ellis, A., & Muñoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning, 5*(1), 1–36.
Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: The Open University.
Metzger, W. (2006). *Laws of seeing*. Cambridge: MIT Press.
Mulligan, J. & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33–49.
Neisser, U. (1976). *Cognitive psychology.* New York: Meredith.
Orton, A. & Orton, J. (1999). Pattern and the approach to algebra. In A. Orton (Ed.), *Patterns in the teaching and learning of mathematics* (pp. 104–123). London: Cassell.
Piaget, J. (1987). *Possibility and necessity.* Minneapolis: University of Minnesota Press.
Resnik, M. (1997). *Mathematics as a science of patterns*. Oxford: Oxford University Press.
Rivera, F. (2007). Visualizing as a mathematical way of knowing: understanding figural generalization. *Mathematics Teacher, 101*(1), 69–75.
Rivera, F. (2008). On the pitfalls of abduction: Complicities and complexities in patterning activity. *For the Learning of Mathematics, 28*(1), 17–25.
Rivera, F. (2009). *Visuoalphanumeric mechanisms in pattern generalization*. Paper presented at the International Meeting on Patterns. Portugal: Viana do Castelo.
Rivera, F. & Becker, J. (2007). Abductive–inductive (generalization) strategies of preservice elementary majors on patterns in algebra. *Journal of Mathematical Behavior, 26*(2), 140–155.
Rivera, F., & Becker, J. (Eds.) (2008a). From patterns to algebra. *ZDM*, *40*(1).
Rivera, F. & Becker, J. (2008b). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM, 40*(1), 65–82.
Rivera, F. & Becker, J. (2009a). Algebraic reasoning through patterns. *Mathematics Teaching in the Middle School, 15*(4), 212–221.
Rivera, F. & Becker, J. (2009b). Formation of generalization in patterning activity. In J. Cai & E. Knuth (Eds.), *Advances in mathematics education*. Netherlands: Springer. in press.
Sophian, C. (2007). *The origins of mathematical knowledge in childhood*. New York: Erlbaum.
Stacey, K. & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 141–154). Dordrecht: Kluwer.
Steele, D. & Johanning, D. (2004). A schematic-theoretic view of problem solving and development of algebraic thinking. *Educational Studies in Mathematics, 57*, 65–90.
Trudeau, J. & Dixon, J. (2007). Embodiment and abstraction: Actions create relational representations. *Psychonomic Bulletin & Review, 14*(5), 994–1000.
Warren, E. & Cooper, T. (2008). Generalizing the pattern rule for visual growth patterns: actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*, 171–185.
Yanos, P. & Hopper, K. (2008). On "false, collusive objectification:" Becoming attuned to self-censorhip, performance and interviewer biases in qualitative interviewing. *International Journal Social Research Methodology, 11*(3), 229–237.
Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Using visual and analytic strategies: a study of students' understanding of permutation and symmetry groups. *Journal for Research in Mathematics Education, 27* (4), 435–457.