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Visualizing as a Mathematical Way of Knowing: Understanding Figural Generalization

In the following passage, Willi Dörfler claims that if we want to find out what students know about, say, patterns, we first need to determine how they see and understand patterns:

> The issue for mathematics education . . . is what does it mean to know something about mathematical objects and how does the learner develop or construct that knowledge? The answer to the question will to some extent depend on the ontological and epistemological status that is ascribed to those mathematical objects. (Dörfler 2002, p. 337)

Often with patterns such as the ones presented in **figures 1** and **4**, teachers assume that there is only one way of producing a generalization that is *algebraically useful,* that is, one that leads to a general formula (Lee 1996). However, results of the interviews that I conducted with ninth-grade students on the pattern sequence in **figure 1** show that there are at least five possible ways (Becker and Rivera 2005). Indeed, Dörfler (2002) asserts that how students know (i.e., epistemology) and what they perceive (i.e., ontology) contributes significantly to the manner in which they develop or construct their knowledge of a mathematical object. The term *mathematical object* encompasses a variety of conceptual entities such as concepts, images, definitions, theorems, figures, diagrams, patterns, and so on. In this article, we focus on patterns of figural objects (or figural cues). The term *figural objects* refers to objects that possess both spatial properties and conceptual qualities (Fischbein 1993). For example, the objects in **figure 1** are figural cues, for they have been constructed in a particular way. Extending them involves inducing an invariant structural property that is evident from one cue to the next, and the goal of induction is to express the property as a valid algebraic generalization.

I interviewed eleven male and eleven female ninth-grade students in a beginning algebra course in a public school in Northern California in order to assess how they actually established a generalization for the tiling squares pattern (**fig. 1**). The motivation behind the interview was a troubling finding from a five-year, districtwide, open-ended assessment involving patterns and functions: While 70 percent of ninth graders tested could extend patterns one by one, less than 15 percent of them could develop an algebraic generalization in closed form (Becker and Rivera 2005). Since 2003, about 60,000

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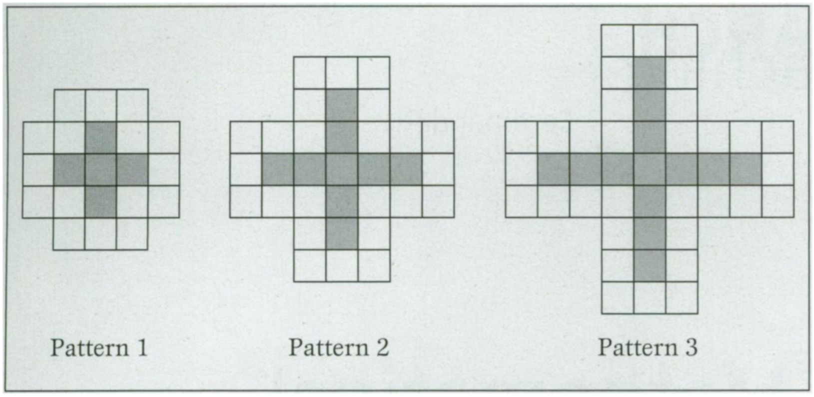


Fig. 1 Tiling Squares problem. From Rivera and Becker 2005

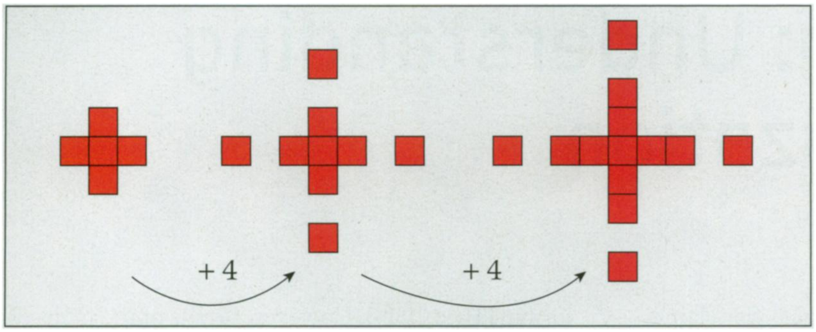


Fig. 2 Additive property for the Tiling Squares problem

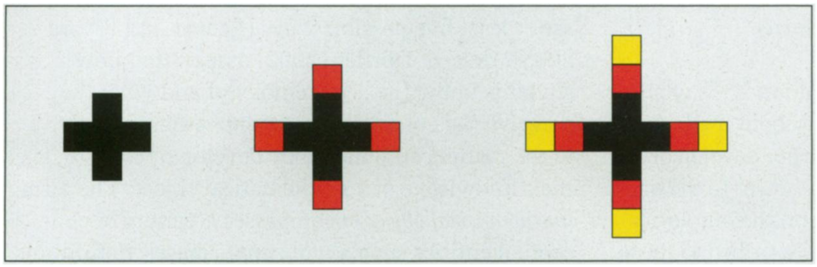


Fig. 3 Concentric visual counting for the Tiling Squares problem

students have participated in the assessment. Hence, it would be interesting to develop strategies that would help students see a connection between the particular and the general in a generalizing activity. That is, what are some strategies that enable ninth-grade students in a beginning algebra course to see through the particulars so that they will eventually develop algebraically useful generalizations?

Keeping in mind Dörfler's (2002) claim in the quotation above, there is a need to articulate epistemological and ontological factors that mutually determine ninth-grade students' ability to establish a generalization for particular patterns. In this article, I first focus on visualization strategies that the students in the interview employed as they attempted to obtain a generalization for the Tiling Squares problem. Then I explore figural-based strategies for teaching algebraic generalization and illustrate them using the written work of students across grade levels. In the conclusion, I briefly discuss benefits that high school students are likely to gain from knowing figural generalization strategies in their developing mathematical understanding of patterns.

FIGURAL ADDITIVE STRATEGY

From interviews with the ninth-grade students, the figural additive strategy seems to be the stepping-stone visual strategy that students working in a visual mode would first employ in expressing a generality. What these students saw was an additive growth in the figural sequence. In fact, it was this invariant property that enabled them to connect one figural cue to the next (see **fig. 2** for an illustration of this strategy). Unfortunately, some of them immediately concluded that the formula $n + 4$ was sufficient to describe the pattern for the number of black tiles corresponding to the recursive formula $a_{n+1} = a_n + 4$. Also, those who used this strategy alone frequently employed unit counting on each figure. For example, pattern 1 in **figure 2** would be counted as 1, 2, 3, 4, 5; pattern 2 would be 1, 2, 3, 4, 5, 6, 7, 8, 9; and so on.

TWO FIGURAL MULTIPLICATIVE STRATEGIES

Counting by "Sides"

Some students saw symmetry among the figural cues. For example, Edward (all student names used in this article are pseudonyms) counted each "side" of the black square tile pattern and multiplied by 4, since there were four arms altogether:

> I looked at pattern 3 and I saw the . . . three tiles that are on each side so I thought I looked at pattern 2 and it just added one so I multiplied four times four with all the sides and just added one in the middle [for pattern 4].

This figural multiplicative strategy suggests the following formula: $n \times 4 + 1$, where n means pattern number. Edward and those students who figurally multiplied did not employ unit counting of tiles, since they were thinking in multiples of one "side" (or "pillar" or "arm") of a figural cue.

Concentric Visual Counting

Some students employed concentric visual counting. That is, they saw the black square tiles in pattern n as embedded in the black tiles in pattern $n + 1$. This strategy suggests the following formula: $(n - 1) \times 4 + 5$, where n means pattern number. The formula is illustrated in **figure 3**. Note that the coefficient 4 in the formula represents the repeated addition of one square on each arm beginning with pattern 2.

PUTTING VISUALIZATION STRATEGIES BACK IN THE LEARNING OF BASIC ALGEBRA

While it was evident that figural strategies were used by some of the ninth-grade students I interviewed, the most favored strategy was numerical (e.g., using the method of finite differences, using trial and error). Students who used a numerical strategy became exclusively occupied with manipulating numbers and testing them against a formula that had been derived numerically. Consider Jennifer's construction of the formula $4n + 1$ using a numerical guess-and-check strategy:

> I started off with more like 2 and that didn't work so then I tried to make 5 work and I did the same thing with 2, 3 . . . and then when I tried it with 4, and I tried to figure a number to make 5 so I add 1, and I tried it on 2 and it still gave me the number.

Even though she was able to come up with the correct formula, she was not able to explain what the numbers 4 and 1 stood for in her formula. If we compare Jennifer's numerical method with Edward's visual construction in the following passage, we can certainly justify the usefulness and the power of visualization in establishing meaningful algebraic generalizations:

> [To find the number of black tiles for pattern 10] I counted the tiles going out, um, so it would be 10. So 10 times 4 'cause [there are] 4 sides. It's um 40 plus 1 for the one in the middle, so that's 41.

For Edward, all the symbols leading to the formula $4n + 1$ have been associated with meanings that have a direct correspondence to the figural cues that produced them.

So how do we promote algebraic generalization via the visualization route? First, we need to remind ourselves that patterns as mathematical objects are not everyday objects. The difficulty faced by many students who struggle to form generalizations is that, for them, patterns involving figural cues appear to be mere drawings. Thus, the ability to notice effectively and to see an algebraically useful pattern from the figures will have to be developed first.

Duval's (1998) theory of apprehending a figure in geometry is useful in talking about figural cues in algebra. Learners apprehend figural cues in two ways: perceptually and discursively. On the one hand, when students such as Jennifer are only able to apprehend figural cues perceptually, they see the cues as primarily consisting of objects (e.g., squares) that continually increase or decrease by a fixed amount and nothing more. One subtle indication of this perceptual apprehension is when students shift their attention away from the visual cues and begin to focus exclusively on the corresponding numerical cues in order to establish a generalization. On the other hand, when students such as Edward are able to apprehend figural cues

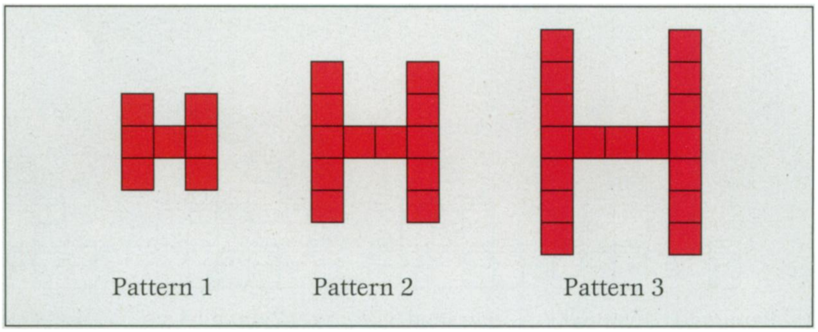


Fig. 4 Tile H pattern. From Roebuck 2005

1. Describe what you see.
2. How is the pattern growing?
3. As each figure grows from pattern to pattern, what changes and what stays the same?
4. How many tiles stay the same from one pattern to the next? Show this. (This refers to the constant b in the linear form $y = mx + b$.)
5. How many tiles have been added from pattern 1 to pattern 2? From pattern 2 to pattern 3? How many tiles would be added in the case of pattern 4? Show this.
6. Is there a relationship between pattern number and the number of tiles that has been added?
7. How many tiles have been added in pattern 2? Do you see multiples of this number of added tiles in pattern 3? In pattern 4? Show this. (This refers to the coefficient m in the linear form $y = mx + b$.)
8. If P represents pattern number and S represents the total number of tiles for any given P, find a direct formula that relates S and P. Explain what your formula means from the figures.

Fig. 5 Guide questions for the tile H pattern

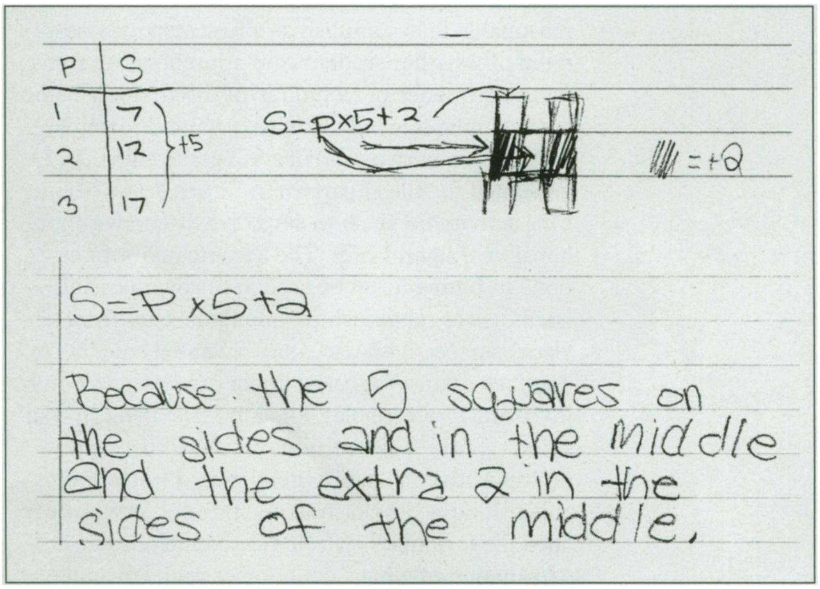


Fig. 6 Written work of Dan and Marlisha on the tile H pattern

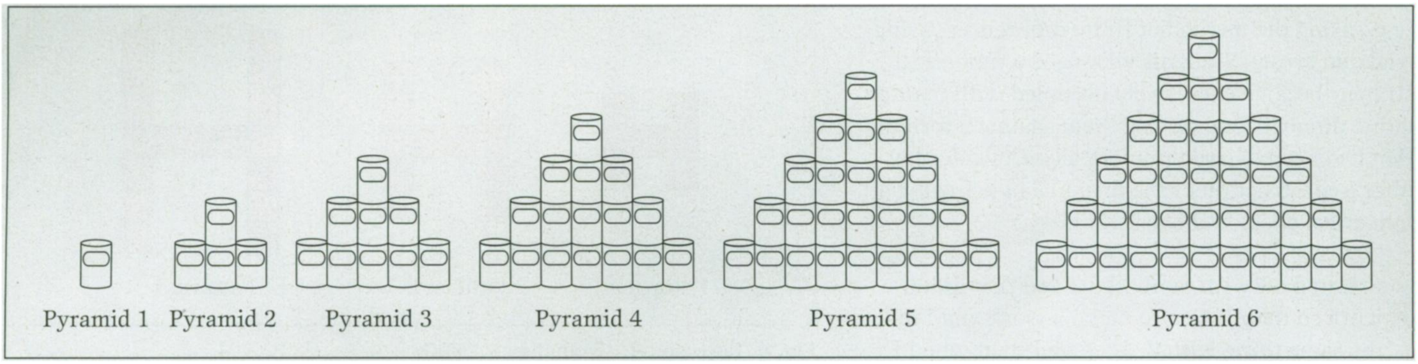


Fig. 7 Pyramid cans pattern. From Sasman, Olivier, and Linchevski 1999

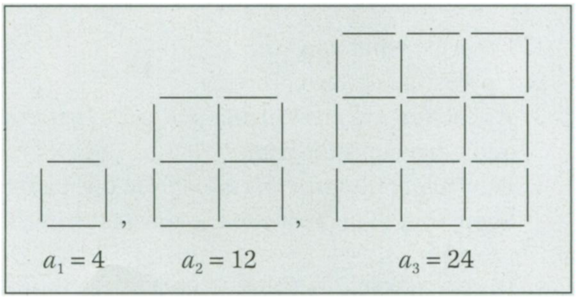


Fig. 8 Array of sticks pattern. From Billstein, Libeskind, and Lott 2007

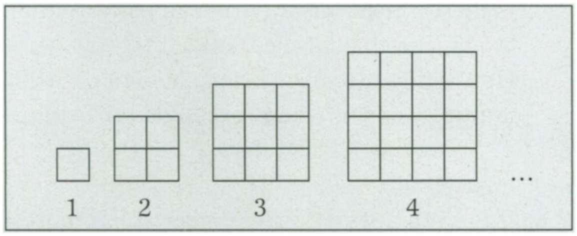


Fig. 9 Transformed pyramid cans pattern

discursively, they see the cues—either individually or in relation to one another—as a configuration of objects (e.g., squares) that are related by some invariant attribute or property. Thus, it seems reasonable to accomplish as a first step an assessment of whether students are apprehending a given figural sequence perceptually or discursively before we ask them to come up with a general formula.

Working with a class of sixth-grade students, I presented the tile H pattern in **figure 4** as a beginning activity for them to develop a discursive apprehension of figural cues. The accompanying questions in **figure 5** can be used to uncover possible attributes or relationships among the cues from a visual perspective. Also, since I wanted students to establish a function-based direct (or closed) formula involving two variables, I had to assist them in transitioning from a figural additive mode to a figural multiplicative mode of generalizing. **Figure 6** presents solutions of two students in the class who developed the formula $S = P \times 5 + 2$ and includes their explanation of what each number in the formula meant. In both of their solutions, Dan and Marlisha saw two properties that stayed the same: (1) the two corner squares in the middle row; (2) the growing multiples of five squares from one cue to the next.

Even if students are able to acquire the ability to apprehend figural cues discursively, we need to take into account the complexity of the cues. Sasman, Olivier, and Linchevski (1999) make a distinction between *transparent* and *nontransparent* figural sequences. In the case of transparent figural cues, such as the tile H pattern, students can determine the appropriate function rules because they are embodied in the structure of the figures, which cannot be claimed in the case of nontransparent figures. For example, the figural sequence in **figure 7** is nontransparent. Also, function rules for some transparent figural sequences (such as the tile H pattern) can be obtained rather easily, while the task can be daunting in the case of more complicated figural sequences, such as the array of sticks pattern in **figure 8**. In the case of nontransparent sequences, something more needs to be done before students are able to *see* a possible function rule from the available cues.

One possible suggestion in dealing with nontransparent figural patterns is to encourage students to further manipulate or transform figural cues into simpler, recognizable forms. When students are able to perform such transformations either visually or mentally on a figural sequence, they manifest a figural change in their apprehension of the figures (Duval 1998). For example, the pyramid cans in **figure 7** can be moved, reorganized, and transformed into a figural sequence of $n \times n$ unit squares (see **fig. 9**).

Another way of dealing with some nontransparent patterns involves a symmetrical counting process. To encourage symmetrical counting, ask students first to look for any presence of symmetry. Then ask them to perform a counting action on one part, A, of a figural cue and apply the same action on those relevant parts of the cue that have the same characteristic(s) as A. It is interesting to think that there is harmony and beauty within a given figural sequence; thus, symmetry can be present as well and should be exploited in order to avoid unnecessary repetitive counting (Zeitz 1999, p. 70).

Examples of symmetrical counting solutions are shown in **figures 10**, **11**, and **12**, which illustrate the visual reasoning of three prospective K–8 teachers on the array of sticks pattern (**fig. 8**). Amanda's work (**fig. 10**) reveals seeing cues that have a configuration of three symmetries: horizontal sticks, vertical sticks, and outside sticks. Kevin initially saw figural cues that consisted of rows and columns of matchsticks (**fig. 11**). Hence, pattern 1 has one row with 2 vertical matchsticks and one column with 2 horizontal matchsticks, pattern 2 has two rows with 3 vertical matchsticks and two columns with 3 horizontal matchsticks, and so on. Martha also saw rows and columns of matchsticks as Kevin did, but she counted in a different way (**fig. 12**). For her, pattern 1 has two rows with 1 horizontal matchstick per row and two columns with 1 vertical matchstick per column, pattern 2 has three rows with 2 horizontal matchsticks per row and three columns with 2 vertical matchsticks per column, and so on.

CONCLUSION: FROM VISUALIZATION TO ALGEBRAIC GENERALIZATION

> When we focus on visualization, we are facing a strong discrepancy between the common way to see the figures, generally in an iconic way, and the mathematical way they are expected to be looked at. There are many ways of "seeing." (Duval 2006, p. 115)

Principles and Standards for School Mathematics (NCTM 2000) strongly recommends incorporating visualization strategies in students' mathematical experiences across content areas. But this is not an easy resolve, since, as Duval (2006) points out in the quotation above, students' acts of noticing and observing patterns may not even be mathematical but may instead follow conventional practices. Further, the difficulty with *seeing* is complicated by the fact that some students have already developed the misconception that mathematics is all about manipulating numbers and numerical expressions and applying algorithms—a misconception that could make the task of visualizing tenuous and rather unnecessary for them. In particular, in the interviews that I conducted with the twenty-two ninth-graders, many of those who employed numerical methods for generalizing were able to establish a formula for the tiling squares pattern from available numerical cues. Hence, they saw no need to visualize. However, they were unable to justify the formula and its parts.

So what benefits do students derive from developing a good visual comprehension of patterns? First, establishing patterns figurally would encourage students to see the dynamic (versus static) component of conceptual construction of mathematical objects and concepts, in particular, pattern objects (designs) in daily life. For example, a visual understanding of linear pattern formation would help them understand—and, one hopes, appreciate—the important role of the slope m in the formula $y = mx + b$, including the implications of the restricted domain (whole numbers) in the corresponding graph of the pattern. Illustrative examples are presented in **figure 13**. In the first task, students consider the effects of the changing values of m, the slope, in forming pattern sequences that either grow (or increase, $m > 0$), decay (or

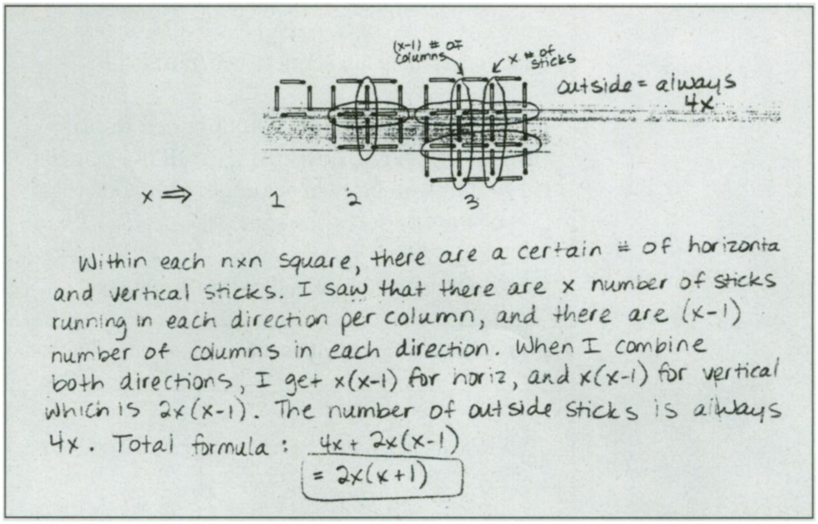


Fig. 10 Visual reasoning of Amanda on the array of sticks pattern

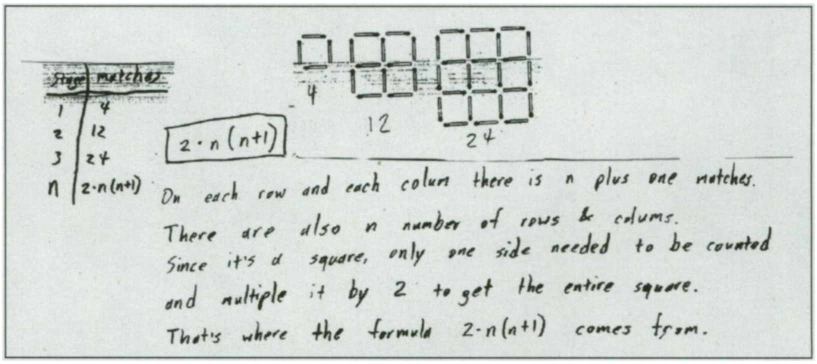


Fig. 11 Visual reasoning of Kevin on the array of sticks pattern

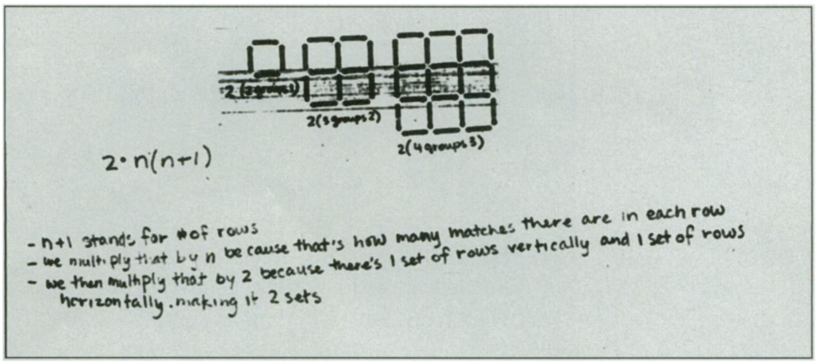
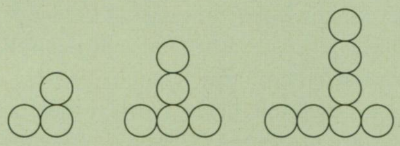


Fig. 12 Visual reasoning of Martha on the array of sticks pattern

1.(a) The pattern sequence of circles below has a growth rate of 2 (i.e., $m = 2$). Explain why it is so. Also, describe the pattern algebraically (i.e., find a direct formula).



1 2 3

(b) Construct a pattern sequence of squares (having at least three cases) of the form $y = mx + b$, where (i) $m > 0$, (ii) $m < 0$, and (iii) $m = 0$. Assume $m, x, y,$ and b are all integers. Also, describe each sequence algebraically.

(c) Based on the results you obtained in tasks (a) and (b) above, how can you tell if a numerical or a figural pattern sequence is increasing, is decreasing, or stays the same?

(d) Construct a table of values for the pattern sequence of circles given in (a) above and then graph. What labels would correspond to the first column or the x-axis? the second column or the y-axis? Describe your graph and state the domain and range.

(e) Construct a table of values for the pattern sequence of squares you obtained in item (b)(ii) above and then graph. Label your first column values and the x-axis. Do the same for the second column values and the y-axis. Describe your graph and state the domain and range.

2. The following problem requires some artistic imagination: Suppose a certain design company has asked you to design square tile patterns, with some constraints. They want two kinds of linear pattern sequences of square tiles that obey the rule $y = mx + b$, where $m, x, y,$ and b are all positive integers and where the first type has an even slope and the second type has an odd slope. Do it now. Next, compare the two pattern sequences you created. Make at least two observations.

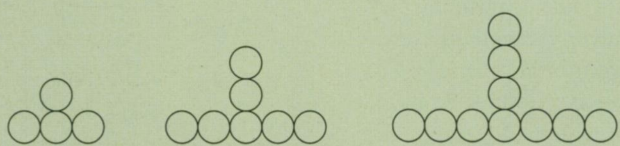
Fig. 13 Visualization activities that explore slopes in relation to pattern construction

1. Below are two formulas for a pattern sequence of squares formed from matchsticks.

$$M = 3P + 1$$
$$M = 4P - (P - 1),$$

where P means pattern number and M refers to the total number of matchsticks. Show that these formulas describe the same pattern.

2. Fill in the table below.



Pattern 1 Pattern 2 Pattern 3

Pattern Number (P)	1	2	3	4	5	6	10	20	117
Total Number of Circles (T)									

3. Three students came up with the following formulas for the table above:

Angelina: $T = (2P + 1) + P$
Rhea: $T = 3(P + 1) - 2$
Maria: $T = (4P + 1) - P$

(a) Explain how each student arrived at her formula.
(b) Rochelle came up with the following formula: $T = 3P + 1$. How might she be thinking about her formula?
(c) Which student's formula is correct? Explain.

Fig. 14 Additional visualization activities that target equivalence decrease, $m < 0$), or remain fixed (or constant, $m = 0$). In the second task, students explore the symmetry design of figural sequences by investigating the effect of the slope m in pattern construction. As an aside, I would strongly recommend that the tasks be explored in a group setting using manipulatives such as colored square tiles, circle chips, pattern blocks, or unit cubes.

Second, generalizing visually oftentimes produces several different formulas for the same pattern, which could not be accomplished if generalizing were done using a numerical method. Such a productive situation naturally extends the discussion to the meaning of equivalent expressions and formulas in mathematics, including why such a notion exists in the first place. Examples of tasks that focus on equivalence are presented in **figure 14**. Students visually explore the equivalence concept either by setting up figural cues (task 1) or by analyzing a given figural sequence (task 2).

Finally, fostering visualization in school algebra articulates the most important description we have about algebra in contemporary times—that is, algebra as the symbolic medium that provides the systematic means to establishing, constructing, and justifying invariant structures and relationships among mathematical objects. Such a medium, by institutional practice and as a consequence of its historical evolution, has been narrowly interpreted in our classrooms as being primarily about manipulating variables and expressions. Visualization in algebra offers an alternative way to understand structures and relationships that necessitate the use of variables.

A unifying thread connecting all the generalizing activities in this article is the use of algebra as a tool for expressing relationships or for finding invariant structures across cues by enhancing the learning of visualization strategies at the same time.

Preparation of this article was supported by a grant from the National Science Foundation, grant #0448649. The opinions expressed do not necessarily reflect the views of the foundation.

Also, the analysis of the interviews with the ninth-grade students that is cited in several locations in this article was conducted with J. R. Becker.

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