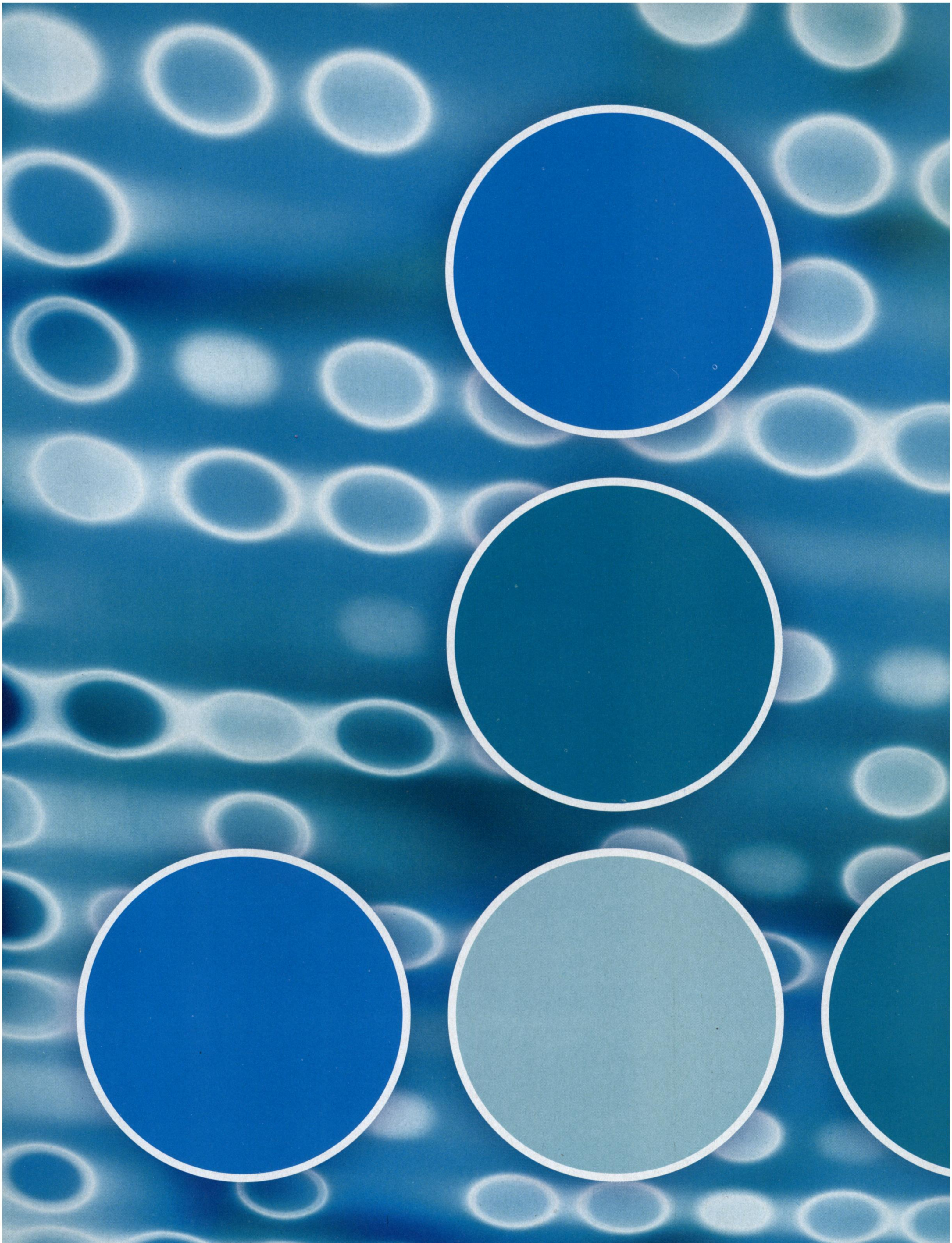

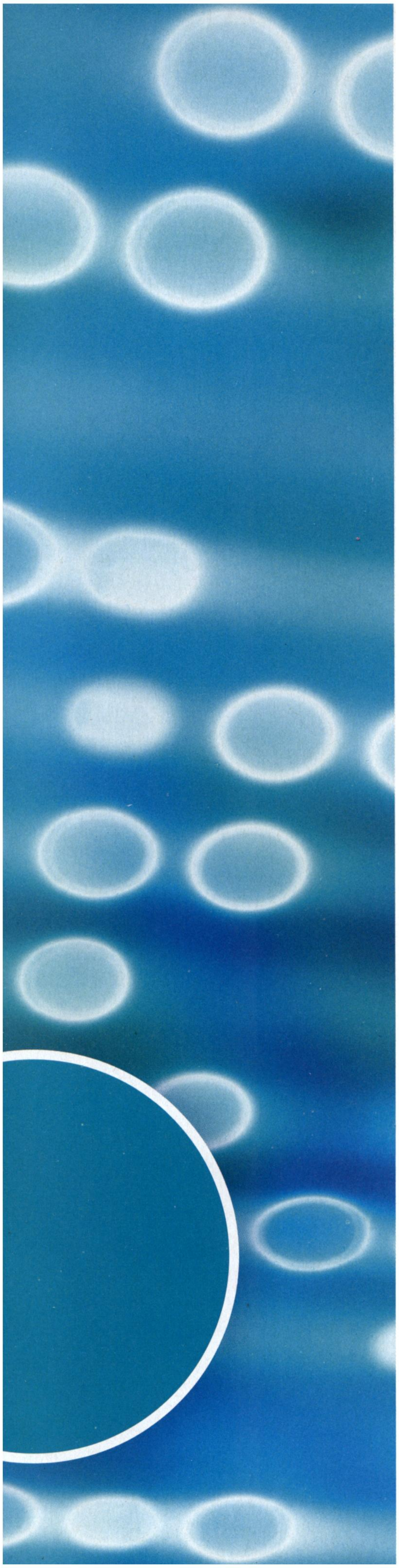

F. D. Rivera and
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ALGEBRAIC REASONING THROUGH PATTERNS

Findings, insights, and issues drawn from a three-year study on patterns are intended to help teach prealgebra and algebra.

Exploring students' performance on patterning tasks involving prealgebra and algebra produced some troubling findings, which we will describe in this article.

Both the qualitative information (student work) and quantitative data (grade-level scores) that we obtained and analyzed from 1998 to 2006 through the Mathematics Assessment Resource Service involved thirty school districts in northern California. Less than 20 percent of the students tested were proficient in establishing an algebraic generalization for figural patterns, such as that in **figure 1**. By algebraic generalization of a pattern, we refer to this explanation by Radford (2008, p. 115):

> [Students] capability of grasping a commonality noticed on some particulars (in a sequence); extending or generalizing this commonality to all subsequent terms; and being able to use the commonality to provide a direct expression of any term of the sequence.

We extend Radford's (2008) comments to include the necessity of justification at the middle school level. In other words, students have to provide some kind of explanation that their algebraic generalization is valid by a visual demonstration that provides insights into why they think their generalization is true (cf. Knuth 2002, p. 488). We use the term *pattern generalization* to convey both mathematical practices of construction and justification of direct formulas.

In year 1, a patterning activity was used to strengthen the students' proficiency with operations on integers and basic equation solving, with a beginning emphasis on increasing patterns. In year 2, a patterning activity was extended to both increasing and decreasing patterns. In year 3, students learned about polynomial expressions and operations, up to multiplication. The patterning activity then focused on the construction of equivalent direct formulas and later provided the context in which to discuss functions and relevant concepts (domain, range, graph) and processes (linear function modeling).

In the remainder of the article, we take a more global view of the three-year study on pattern generalization and focus on what we learned with an eye to describing the mathematical content knowledge for teaching (MKT). Stylianides and Ball point out that MKT pertains to those "particular forms of mathematical knowledge that is useful for, and usable in, the work that teachers do as they teach mathematics to their students" (2008, p. 308). The content and suggestions that we pursue here have been rooted in our experiences with typical students that most teachers encounter daily in their classes.

Fig. 1 The circles pattern

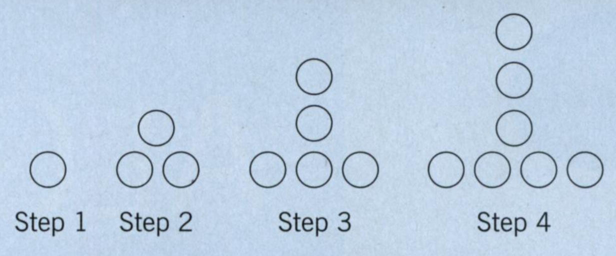


Fig. 2 Generalizations made by Shawna and Dung on the circles pattern

You are now going to write a message to an imaginary Grade 6 student clearly explaining what she or he must do in order to find out how many circles there are in any given figure of the sequence.

Shawna: First you check the figure number, which is the bottom row. Then the vertical you minus one and then you add them together. Then the answer is what you got.

Dung: You can find out how many pieces are in any figure by looking at what number figure it is. Then on the bottom row it should have how many spaces the figure number is. The top column should be one less than the figure number.

Find a formula to calculate the number of circles in the figure number "n."

Dung: Figure n = bottom row pieces. N is how many numbers are on the bottom.
Column = $n - 1$
Figure $N + n - 1$ = how many pieces.

LIMITED-ACCESS PATTERN ISSUES

Think for a moment about how we know when our students are at the crucial stage of generalizing in patterning activity. Perhaps we have the view that generalizing and inductive reasoning fall under the same category. Dreyfus, for instance, characterizes generalizing as "deriving or inducing from particulars, identifying commonalities, and expanding domains of validity to include large classes of cases" (1991, p. 35). Why do such actions appear to be easier said than done? In addition, pattern generalizing on the basis of three or four initial stages can be problematic. For example, different ways of extending the sequence {1, 2, 4, . . .} would influence the content of the corresponding algebraic generalization.

Students who have no formal experiences in the symbolic system of algebra 1 are naturally expected to express their generalizations in words. Take, for example, the prealgebraic mathematical proficiency of Shawna and Dung, sixth graders in year 1, who are given the pattern in **figure 1**, and whose responses are in **figure 2**. Before taking part in a teaching experiment on generalization, they showed basic competence in whole-number operations and relative success in evaluating simple algebraic expressions (e.g., if $a = 2$ and $b = 3$, then $2a + b = ?$). When they were asked to write a message that conveyed their generalization for the circles pattern, their generalizations involved words or a combination of words and variables. Although Shawna used words, Dung's message contained the variable n that made sense only within the context of his entire response. Both students, however, were quite close to an algebraic generalization so that formal

Some Background on Our Study

This study explores students' multirepresentational practices when they develop an algebraic generalization for pattern sequences that have either numerical or figural stages. In all three years of the study, the second author conducted the pre- and postclinical interviews with the students. The first author designed and implemented classroom teaching experiments on pattern generalization while working with classroom teachers.

Each teaching experiment lasted twelve consecutive weeks and often occurred within the context of a big mathematical idea or unifying strand. **Table 1** shows the general content flow that allowed the first author to pursue pattern generalization in all three years of the study in light of the existing constraints of the school (state standards, department and classroom practices, pacing guides, benchmark assessments, and so on).

Part of the success in implementing the teaching experiments resulted from the regular weekly planning time that occurred between the first author and the classroom teachers. Often, the discussion involved findings drawn from ongoing assessments and ways to further refine the lessons so that students achieve the learning goals and objectives of the experiments. The original group of student-participants was in the same class together only during years 1 and 2 of the study. In year 3, fifteen were kept together and mixed with a group of seventh and eighth graders. The new group then pursued the recommended California standards for algebra 1.

A majority of the student-participants in both groups were Asian Americans; a small number were Caucasian, African American, and Hispanic. We also note that the beginning mathematical proficiency levels of the students in the original group fell under the basic and below categories. In year 3, although all the students were proficient and above in prealgebra, a preassessment clearly showed significant differences in terms of their ability to generalize patterns in favor of the old group. Considering the preassessment findings and anticipating the conceptual requirements of formula construction, the first author and the grade 8 teacher decided to ground the students' initial common algebraic experiences on polynomials and polynomial operations, which actually made sense in light of their experiences with integers and integer operations.

Table 1 Sequence of content in relation to the teaching experiment on patterning and generalization

Stages	Before Teaching Experiment	Actual Teaching Experiment	After Teaching Experiment
Year 1 (sixth-grade math)	Integers and integer operations using binary chips, properties of integers	Linear pattern generalization (increasing patterns)	Point plotting, rational numbers
Year 2 (seventh-grade prealgebra)	Integer and integer operations on a number nine, properties of integers	Linear pattern generalization (increasing and decreasing patterns	Rational numbers, positive and negative slopes, graphing lines
Year 3 (eighth-grade algebra 1)	Integers and exponents, polynomials and polynomial operations and properties (up to multiplication) using AlgeBlocks®	Linear pattern generalization, construction and justification of equivalent direct formulas	Functions (domain, range, graph), linear functions, slopes (positive and negative, zero, undefined, fractional)

instruction involving variables to express relationships was apt to be fruitful.

Figure 3 shows the pattern generalization of Jenna, sixth grader in year

Fig. 3 Jenna's extension and generalization of the circles pattern

You are now going to write a message to an imaginary Grade 6 student clearly explaining what she or he must do in order to find out how many circles there are in any given figure of the sequence.

Jenna: By using the pattern and the pattern is (on top) 0, 1, 2, 3.

1, in relation to the circles pattern. She imagined her pattern to be increasing in a particular way. Specifically, she saw that the number of circles on each row in the pattern corresponded to the step number and then assumed that the number of column-circles oscillated from 0 to 3 over a cycle of four steps. She then used the same structure in extending the pattern from stages 5 through 8. Her written message reflected what she saw and interpreted to be her generalization of the pattern. It is an acceptable response, despite the fact that it is difficult to establish a direct formula.

We assume that when we ask our students to generalize, the result has

Fig. 4 A contextual problem allows students to use arrow strings before generalization rules are expressed.

John had \$37 before he earned \$10 for delivering newspapers one Monday. The same day, he spent \$2 for an ice cream cone. Tuesday, he visited his grandmother and earned \$5 for washing her car. Wednesday, he earned \$5 for baby sitting. On Friday, he spent \$2.75 for a hamburger and fries and \$3 for a magazine.

A typical arrow-string solution of this problem is shown below.

$$\$37 \xrightarrow{+\$10} \$47 \xrightarrow{-\$2} \$45 \xrightarrow{+\$5} \$50 \xrightarrow{+\$5} \$55 \xrightarrow{-\$2.75} \$52.25 \xrightarrow{-\$3} \$49.25$$

Source: MiC Development Team 2003, p. 3

Fig. 5 A contextual problem with an invariant relationship can lead students toward general, and ultimately algebraic, descriptions of a relationship

The Rainbow Cab Company charged fares based on the distance traveled. Also, the starting amount was \$2 and the price for each mile was \$2.50. Decide and justify which arrow formula would yield the correct fare.

$$\text{No. of miles} \xrightarrow{\times \$2.50} \underline{\qquad\qquad} \xrightarrow{+\ \$2.00} \text{total price}$$

$$\$2.00 \xrightarrow{+\ \text{number of miles}} \underline{\qquad\qquad} \xrightarrow{\times \$2.50} \text{total price}$$

$$\text{No. of miles} \xrightarrow{+\ \$2.00} \underline{\qquad\qquad} \xrightarrow{\times \$1.50} \text{total price}$$

Source: Adapted from the MiC Development Team 2003, p. 17

to take the form of a direct formula (i.e., an equation in function form). For example, when Shawna, Dung, and Jenna were asked to generalize the same circles pattern at the end of the teaching experiment in year 1, their responses took the form of $C = n \times 2 - 1$, where C refers to the total number of circles and n the stage number. They justified their formula numerically in the context of a table, which they saw as consisting of ordered pairs whose dependent terms consistently increased by two circles. Frank, sixth grader, justified the same formula when he claimed that he saw each stage in the pattern as "doubling a row and minusing a chip," which we consider to be a type of visual justification.

CONSTRUCTING DIRECT FORMULAS

The initial teaching experiment that helped the year 1 students gain their ability to write direct formulas has been drawn from several algebra units of the Mathematics in Context (MiC) curriculum. For example, in *Expressions and Formulas* (MiC Development Team 2003), arithmetical operations using arrow strings were investigated. Many contextual problems were solved using arrow strings before any kind of generalization was introduced (see **fig. 4**). Then students generalized invariant relationships using arrow strings (see **fig. 5**).

Finally, patterns that initially consisted of figural and numerical stages were given, and students were asked to do the following:

1. Generalize regularities and replace numbers with variables and arrows with the appropriate symbols (operation, equal sign);
2. Establish and justify visually; and
3. Establish generalizations numerically.

Overall, a careful sequencing and discussion of problems such as those in

figures 4 and **5** helped students move to a formal symbolic representation of linear patterns.

In years 1 and 2, the students generalized numerically and visually. In year 2, they learned about the coordinate system that enabled them to make sense of linear patterns graphically. The numerical generalization strategy, which the first group developed in class in year 1, reflected the mathematical practice of common differences. That is, they would initially set up a two-column (or two-row) table. From the table, they would then obtain a common difference among the dependent terms. It conveyed to them the need to set up a direct formula following $A = n \times B + C$, where n represents the stage number in a pattern, A the outcome or total, B the common difference, and C the constant amount needed to completely match the dependent terms.

However, we noticed that as students became more competent in obtaining direct formulas for both increasing and decreasing linear patterns, their ability to justify their formulas weakened over time. Many of them confused justification with construction, using the common difference method. Further, few students could offer different equivalent formulas for the same pattern because the numerical strategy showed them only one route to constructing a direct formula. Among those who consistently generalized visually, construction and justification of a direct expression were seen as mutually related. For example, Frank's visual perception of the general structure of the circles pattern as involving the "doubling of a row and minusing a chip" was his way of justifying his formula $C = n \times 2 - 1$.

In year 3, the first group oriented the new to the common difference method as described above. Both groups also used a visual grouping approach, which is discussed in some detail later in this article. At this stage, it is necessary to address the issue of how it is possible to perform pattern generalization on the basis of an incomplete set of particular stages. We accomplish this through Peirce's (1958) notion of abduction.

In Your Classroom

Students benefit from having both visual and numerical modes of generalizing. An effective way of helping them develop both modes is to let them share and critique one another's generalization strategies. Teachers first need to set up a safe classroom environment in which students are fully aware that they can generate their own solution processes in relation to a task and that sharing their work with others signals ownership of the task. One useful norm that should be frequently discussed in class deals with the possibility that at least one student will always have access to a more sophisticated or a more efficient solution or strategy than the ones that are offered (Rasmussen, Yackel, and King 2003).

An interesting way to facilitate the norm discussed above is to show samples of student work, then allow them to discuss the advantages and disadvantages of each. For example, students can be asked to analyze the work of other students. Different solutions or strategies can be posted in several parts of the classroom. Students visit each part, take notes, and then share their learning either within the context of a group or the entire class.

Closure is an important phase. The teacher should solicit a synthesis of different approaches and offer a different approach, especially when intended content needs to be shared.

PATTERN ASSUMPTIONS

When we infer something about a phenomenon whose totality we can never fully grasp, we go through two complementary stages that Peirce (1958) refers to as *abductive* and *inductive reasoning*. Abductive reasoning involves forming a reasonable hypothesis about the phenomenon. To form that hypothesis, we verify and test the abduced hypothesis several times to see whether it makes sense. When doctors perform an assessment or when jurists analyze a case with incomplete data, the preparatory stage in developing an inference involves abduction. When generalizing a pattern on the basis of a few known initial stages, we need to pay attention to how abduction and induction can be used together to construct and justify a complete and valid algebraic generalization (Rivera 2008). Following Peirce, abduction as a concept exists and is used only in relation to induction. Thus, a complete pattern generalization involves complementary acts of abduction and induction and, of course, justification.

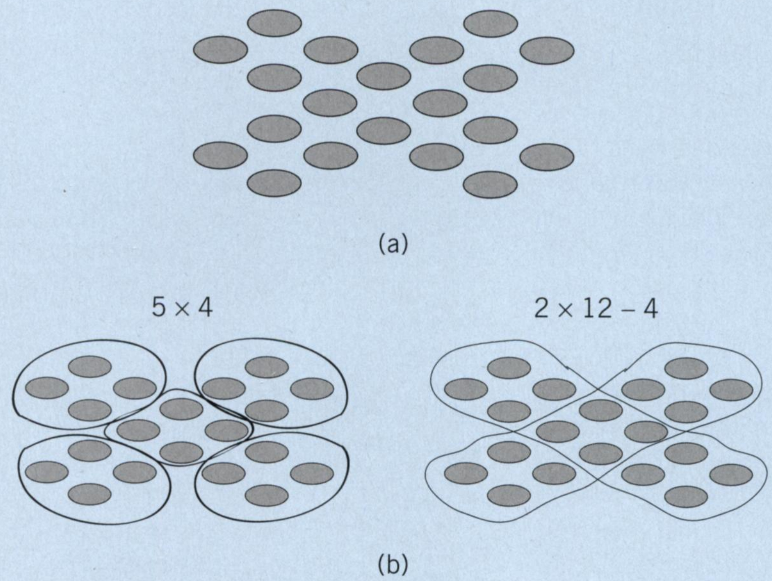
Why Should We Be Interested in Abduction?

Any pattern generalization makes sense but only in the context of an interpreted structure that we initially abduce from the given initial stages. In fact, the extended stages (both near and far generalizations) depend on this necessary abductive step. Induction is used to verify the relevant abduction of the known and extended stages. For example, Shawna, Dung, and Frank abduced linear growth in the case of the circles pattern, whereas Jenna abduced growth in a different way.

From this perspective, mathematicians and scientists work the same way. For example, when mathematicians

Fig. 6 An activity that promotes multiplicative thinking provides the opportunity for various representations.

Find three mathematical expressions that will help you find the total number of circles without counting them one by one. Examples of expressions are 3×4 and $2 \times 5 - 3$. **Explain why each of your expressions makes sense by showing it in separate copy of the same figure below.**



develop a model of a particular phenomenon, they initially make assumptions or hypotheses (abductions) on the basis of what they know about the phenomenon at the time. Then they test them repeatedly (induction) and prove (using rigorous deductive techniques or a computer verification of an extremely large set of cases). Of course, among middle school students, justification is relative to their level of mathematical experiences.

Hence, the first thing we need our students to consider when they are confronted with a patterning task is how to generate viable, acceptable, and reasonable abductions that could potentially lead to an algebraic generalization that could be justified. From our experiences, multiple abductions could lead to multiple generalizations of the same pattern. The three scaffolding questions below help students develop and articulate a possible abductive structure for any given pattern:

1. How might you extend this pattern? Why extend it that way?
2. What stays the same and what changes in your pattern?
3. Is there another way to extend the pattern? How so?

Question 1 provides the necessary initial abductive reflection; question 2 focuses on an explicit articulation of a structure. Question 3 is meant to bring to the surface other plausible, and possibly equivalent, abductive claims. Thus, structure sense and what is eventually generalized are both relative to the abductions they find meaningful to pursue.

GENERALIZING THROUGH COUNTING

In the case of increasing patterns, students often begin with an additive-based generalization. For example, the pre-instructional clinical interview with Shawna in year 1 in relation to the circles pattern indicates that she initially saw the particular stages to be each increasing by two circles. With more scaffolding from the second author (interviewer), her written response in **figure 2** reflected a deeper understanding of an interpreted structure while remaining additive in overall form (i.e., adding the row and column circles).

On the basis of our data, additive thinking in pattern generalization is of two types:

- Type 1: Students initially formulate a surface-based next-to-current relationship, typically expressed in the generic response "add *x*."
- Type 2: Students may be so preoccupied with obtaining the total number of objects per stage number through counting one by one that they fail to notice a possible structure within a pattern stage or among two or more stages.

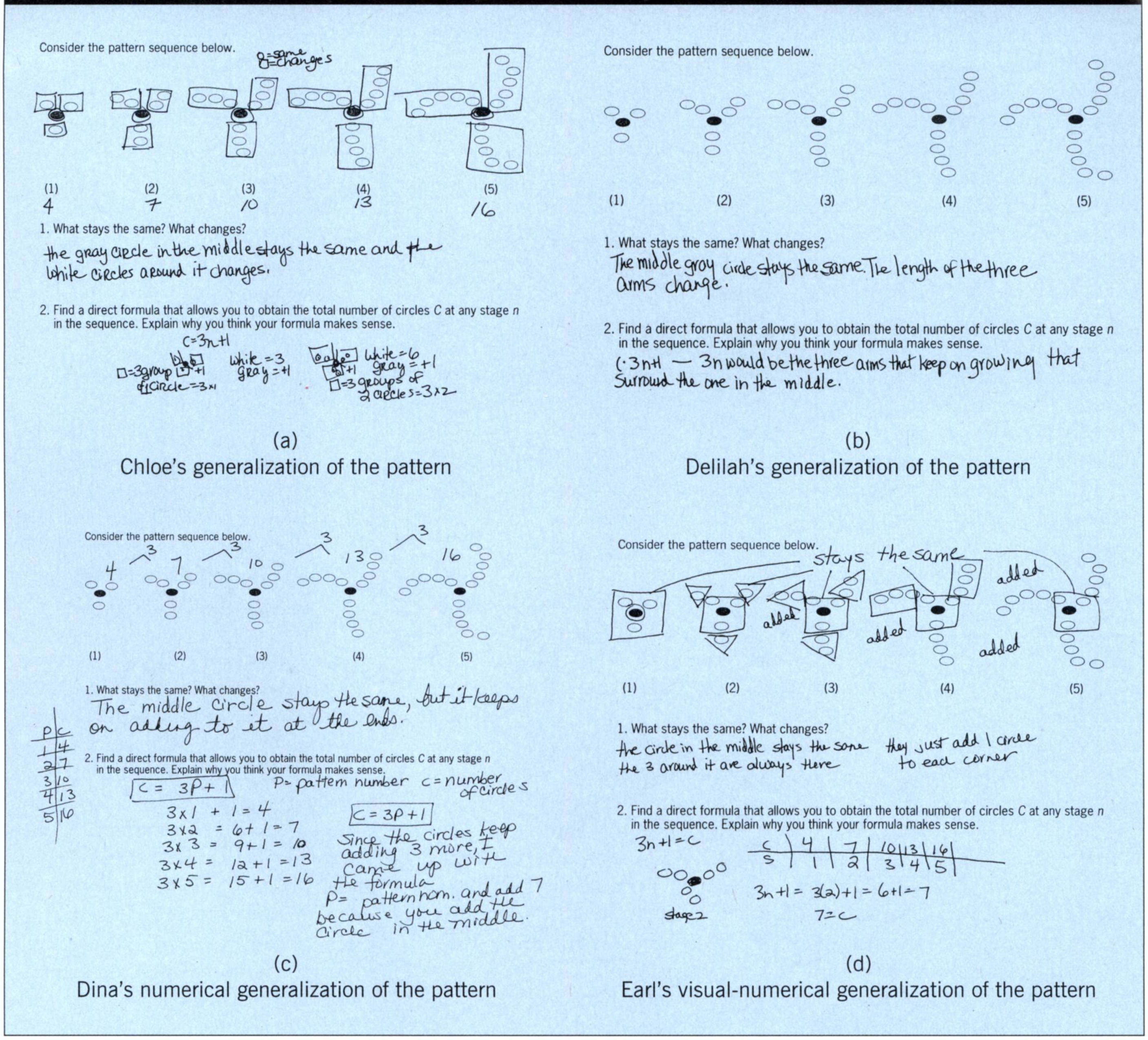
Hence, teachers need to implement activities that explicitly foster counting by groups or, more formally, multiplicative thinking. The place to start is when students learn the concept of multiplication of integers, in which

$$a \bullet b = \underbrace{b + b + b + \cdots + b}_{a \text{ times}}$$

is the same as *a* copies (or groups) of *b*.

In the activity in **figure 6a**, for example, ask students to obtain several different ways of counting the total number of circles and express their answers in multiplicative form. Examples of possible approaches to counting the circles are illustrated in **figure 6b**. A good follow-up activity involves asking students to consis-

Fig. 7 The three-legged circle pattern task in compressed form (the pattern is drawn from Beckmann 2008, p. 496)



(a) Chloe's generalization of the pattern

(b) Delilah's generalization of the pattern

(c) Dina's numerical generalization of the pattern

(d) Earl's visual-numerical generalization of the pattern

tently use the same multiplicative grouping strategy in finding the total number of, say, the circles at every stage in the circles pattern. For example, Chloe applied multiplicative thinking in obtaining the total number of circles at every stage in the three-legged circle pattern. Her pattern generalization in **figure 7a** exemplifies the continuous relationship between formula construction and justification.

GENERALIZING BY SPECIALIZING

We find specializing to be an effective generalizing strategy. In our study, many students often paid attention to the known individual stages in a pattern one at a time that enabled them to abduce a possible overall structure for the pattern (Mason, Burton, and Stacey 1985). Specializing leads to a local generalization, a way of "seeing the general through the particular."

Analogical reasoning in pattern generalization involves applying the specialized observation obtained in a single stage to the remaining known stages, which then becomes the basis in stating an abduced structure across stages, both known and unknown. For example, the generalizations of Shawna and Dung in **figure 2** are results of seeing an analogous-additive relationship among the stages in the circles pattern. Frank's generalization of "doubling a row and minusing a chip" reflects an analogous-multiplicative relationship. The scaffolding questions below help students develop an analogical strategy leading to an algebraic generalization:

1. Focus on the first cue (or any single cue). Describe it for me. Do you see any relationship among the parts of the cue? Do you see any symmetry? Do you see a part of the cue being repeated elsewhere on the cue?
2. Use what you observed in (1) to analyze the second (or another) cue. Do you notice the same structure? How so?
3. If what you found in (1) and (2) appear to be consistent, does it apply to the third (or still another) cue? The fourth (or yet another) cue? If you extend your observation to those cues you do not see, how might the nth cue look?

Delilah made an analogy-based generalization shown in **figure 7b**. She noticed a middle gray circle with "three arms that keep on growing," which explains why her algebraic generalization took the form $C = 3n + 1$.

THE INEVITABLE DUAL CODING REALITY OF GENERALIZING

When patterns appear as sequences of figural stages, such as those in **figures 1** and **7**, some students will establish their generalizations visually, whereas others will do so numerically. Visual generalizers, such as Chloe in **figure 7a** and Delilah in **figure 7b,** perceive meanings from pictures and diagrams of particular stages. Dina, a numerical generalizer, whose work is shown in **figure 7c**, derives satisfaction and comfort in manipulating numbers with trial and error and other numerical patterning techniques (Becker and Rivera 2005). In our study, we also saw that visual generalizers tend to be adept at using a numerical strategy (Rivera and Becker 2008), such as Earl, whose work is shown in **figure 7d**. Earl initially saw stage 1 as the part that stayed the same from the first stage to the next. When it became difficult for him to establish an algebraic generalization visually, he then employed the common difference strategy leading to the formula $3n + 1 = C$, which he verified numerically on two cases and verified visually using stage 2.

Although visual approaches are powerful, we underscore the significance of numerical approaches in generalizing activities that do not involve visual stages (i.e., nonfigural patterns). For example, in eighth grade, students were successful in dealing with function problems such as that below:

The Cost Function Task

The cost to rent a construction crane is $750 per day, plus $250 per hour of use.

a. Set up a table of values showing 5 consecutive hours of use. By default, hour 0 equals $750 (why?).

b. Set up a function rule that expresses the total rental cost T in terms of the number of hours h used per day. Explain your formula.

c. How much does it cost to rent the crane for 8 hours? 12 hours? Explain.

d. How many hours does a construction company have to use a crane for a maximum budget of $2500 per day?

e. What is the domain and range of the given problem? Explain.

When we gave the same and similar problems to students (e.g., textbook tasks involving tables of numerical values), they interpreted the construction and justification of function rules in terms of their earlier experiences in patterning activity.

CONCLUSION

A patterning activity gives students an opportunity to construct and justify an algebraic generalization. Pattern generalization involves a synergy of abduction, induction, and proof, and students must—

1. state their assumptions and hypotheses about a plausible structure of a pattern as they construct a reasonable direct formula (the abductive phase);
2. verify and test their choice of abduction repeatedly over several stages (the inductive phase); and
3. justify (e.g., visually).

Teachers need to emphasize the possibility of having different approaches in visualizing a figural pattern, which leads to multiple and/or equivalent algebraic generalizations. Hence, students need to be strongly encouraged to share their approaches and interpret them, which should deepen their repertoire of analogical and multiplicative strategies for pattern generalization. Teachers also need to support group activities, communication, and explanation of student work to facilitate students' learning new strategies from one another.

REFERENCES

Becker, Joanne Rossi, and Ferdinand Rivera. "Generalization Schemes in Algebra of Beginning High School Students." In *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, edited by Helen Chick and Jill Vincent, vol. 4, pp. 12–28. Melbourne, Australia: University of Melbourne, 2005.

Dreyfus, Tommy. "Advanced Mathematical Thinking Process." In *Advanced Mathematical Thinking*, edited by David Tall, pp. 25–41. Dordrecht, Netherlands: Kluwer, 1991.

Knuth, Eric. "Proof as a Tool for Learning Mathematics." *Mathematics Teacher* 95 (October 2002): 486–90.

Mason, John, Leone Burton, and Kaye Stacey. *Thinking Mathematically.* Wokingham, England: Addison-Wesley Publishing Co., 1985.

Mason, John, Alan Graham, and Sue Johnston-Wilder. *Developing Thinking in Algebra.* London: The Open University, 2005.

Mathematics in Context Development (MiC) Team. *Expressions and Formulas.* Chicago, IL: Holt, Rinehart, and Winston and Encyclopaedia Britannica, 2003.

Peirce, Charles. *Collected Papers of Charles Sanders Peirce,* edited by Charles Hartshorne and Paul Weiss. Cambridge, MA: Harvard University Press, 1958.

Radford, Luis. "Iconicity and Contraction: A Semiotic Investigation of Forms of Algebraic Generalizations of Patterns in Different Context." *ZDM* 40, no. 1 (2008): 83–96.

Rasmussen, Chris, Erna Yackel, and Karen King. "Social and Sociomathematical Norms in the Mathematics Classroom." In *Teaching Mathematics through Problem Solving: Grades 6–12*, edited by Harold Schoen and Randall Charles, pp. 143–55. Reston, VA: National Council of Teachers of Mathematics, 2003.

Rivera, Ferdinand. "On the Pitfalls of Abduction: Complexities and Complicities in Patterning Activity." *For the Learning of Mathematics* 28, no. 1 (2008): 17–25.

Rivera, Ferdinand, and Joanne Rossi Becker. "Middle School Children's Cognitive Perceptions of Constructive and Deconstructive Generalizations Involving Linear Figural Patterns." *ZDM: International Journal in Mathematics Education* 40, no. 1 (2008): 65–82.

Stylianides, Andreas, and Deborah Ball. "Understanding and Describing Mathematical Knowledge for Teaching: Knowledge about Proof for Engaging Students in the Activity of Proving." *Journal of Math Teacher Education* 11 (2008): 307–32.

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