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Abduction–induction (generalization) processes of elementary majors on figural patterns in algebra

F.D. Rivera*, Joanne Rossi Becker

Department of Mathematics, San José State University, United States

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Abstract

The article deals with issues concerning the abductive–inductive reasoning of 42 preservice elementary majors on patterns that consist of figural and numerical cues. We discuss: ways in which the participants develop generalizations about classes of abstract objects; abductive processes they exhibit which support their induction leading to a generalization; ways they justify their generalizations in the abductive stage, and; the effects of figural and numerical cues in the manner they construct a plausible abductive generalization. Two types of abductions are explored, model-based and manipulative. A proposed abductive–inductive reasoning process for pattern sequences is presented and discussed in the concluding section.

Keywords: Abduction; Induction; Generalization; Linear and quadratic patterns; Algebraic thinking

1. Introduction

In mathematics courses for university elementary majors, students learn to reason inductively, deductively, and spatially. Students who learn to reason according to the principles governing each type are expected to develop characteristics associated with *adult cognition*, marked by an ability to see through superficial features in favor of deep structures, attributes, properties, and relations (Gelman & Wellman, 1991). With tasks that involve deduction, for instance, students are expected to correctly assess the validity of arguments (Fig. 1) using known principles such as *modus ponens* and *modus tollens*. However, such tasks can be complicated for some students because a variety of everyday arguments possess some pragmatic truth based on the specificity of context (Holland, Holyoak, Nisbett, & Thagard, 1986). However, being able to reason mathematically by deduction means overcoming the limitations of such context through the use of formal rules of logic. Logic, Kyrburg (2001) astutely points out, is not only about reason, reasoning, and arguments as it is also "concerned with right reason . . . good arguments . . . [and, thus,] contains an ineliminable prescriptive element" (p. 589). A similar claim can be made in the case of spatial reasoning. Students who depend primarily on the external features of a mathematical object alone to develop conjectures may produce statements that contain misconceptions or naïve conceptions about the object. For example, those who are unfamiliar with several possible shapes of a triangle might conclude that an altitude or the orthocenter of any triangle would always lie in the interior of the triangle. Thus, students acquire "spatially sensitive axioms and rules of inference" that sanction good arguments and reasons from faulty ones (Chandrasekaran, 1997).

* Corresponding author at: 1 Washington Square, San José, CA 95192, United States. Tel.: +1 408 9245170; fax: +1 408 9245080.
E-mail addresses: rivera@math.sjsu.edu, becker@math.sjsu.edu (F.D. Rivera).

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If today is Thursday, then our mathematics class meets today.

Today is Thursday.

Therefore, our class meets. (Valid)

If you attend school today, I will give you $10.

You did not attend school today.

Therefore, I will not give you $10. (Invalid)

Fig. 1. Examples of valid and invalid deductive arguments.

This article deals with issues concerning the abductive–inductive reasoning of preservice elementary majors on patterns that consist of figural and/or numerical cues. At the outset, we were interested in assessing how undergraduate students who expressed an interest in teaching K-8 children would solve generalizing tasks such as those that are shown in Figs. 3 and 5. The tasks given to the participants contain sequences of figural and numerical cues which taken together comprise classes of abstract objects.[1] The accompanying questions oftentimes involve a twin calculation–encapsulation process, that is, from determining specific output values to abductively forming a viable general expression which can generate any element in the class. Harel and Tall (1989) define generalization as referring to "the process of applying a given argument in a broader context" (p. 38). In particular, Dreyfus (1991) identifies the task of generalizing as "deriv[ing] or induc[ing] from particulars, identify[ing] commonalities, [and] expand[ing] domains of validity" to include "large classes of cases" (p. 35). Thus, the central purpose of generalizing tasks at the elementary level is to help learners develop an ability to generalize from particular instances and be able to express the generalization in ways that are both meaningful to them and valid from the standpoint of institutional practice. The requirement of an elementary proof that a generalization indeed holds true for all the elements in the largest domain possible can take several possible routes such as justifying it visually if it is accompanied by a given pattern sequence of figural cues. Thus, in this paper, we resolve the following issues:

- How do preservice elementary majors develop generalizations about classes of abstract objects? In particular, what abductive processes do they exhibit that support their induction leading to a generalization?
- How do they justify their generalizations in the abductive stage?
- How do figural and numerical cues influence the manner in which an abductive inference is constructed?

This article has been drawn from a much larger project on algebraic generalization at the K-13 and preservice-teacher levels. The fundamental problem concerns ways in which learners in the area of school algebra acquire their abilities (and not "ability") to generalize. Thus, relevant questions include the following: What abductive processes enable individuals to correctly develop viable generalizations based on a limited and incomplete set of concrete instances? How is it that some are capable of justifying their abductive generalizations, while others are not? Sections 2 and 3 comprise the theoretical framework used in this study, with Section 2 addressing the nature of classes of objects in a generalizing task and Section 3 providing a detailed, albeit not exhaustive, discussion of the concept of abduction. In Sections 4 and 5, we address the bulleted issues above viz-a-viz the three generalizing tasks involving figural patterns which 42 preservice elementary majors solved using several different abductive strategies. Section 6 provides a general discussion and a provisional closure.

2. Classes of objects in a generalizing task

Following Shipley's (1993) characterization of categories in empirical induction, a *class of abstract objects in a generalizing task* consists of the following properties: there is a closed formula that can be derived from the cues; the

[1] Sets of abstract objects such as even and odd numbers and those used in this study possess both perceptual and conceptual properties. At the outset, we assume that generalizing knowledge about sets of abstract objects does not require making a distinction between processes of generalization that focus on the acquisition of either perceptual (sensory) or conceptual (nonsensory) information. Constructing generalizations is grounded on a dynamic relation between percept and concept. Also, Mervis, Johnson, and Scott (1993) clearly point out that even "expert knowledge remains rooted in perception throughout the continuum of knowledge acquisition" (p. 154).

cues, even if incomplete, provide a plausible range of terms that enable the performance of generalization, and; the cues resemble each other in some way. For instance, the sequence $\{2, 4, 6, 8, \ldots\}$ is a class and the closed formula $2n$ is one way of describing the overall structure of each number in the sequence. Also, it is perceptually apparent that evenness is one characteristic that is common to the numbers in the sequence. The Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ is another example of a class that can be generally described by the recursive relation $a_{n+1} = a_{n-1} + a_n$ (where $a_1 = a_2 = 1$). Its closed formula is $(1/\sqrt{5})[((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n]$ with the additional assumption that the numerical cues would obey the stated recursive form. The arithmetic class $\{3, 8, 13, 18, \ldots\}$ can be generalized by the direct expression $5n-2$ under the condition that the class is an increasing sequence and where $n \geq 1$. Resemblance encompasses implicit (deep) and explicit (surface) properties that cues within a class have in common, and these properties are not inherently *a priori*. That is, depending on the knowledge and experiences of learners, they abduce properties by *projecting* them onto individual elements in a class being tested. Projecting involves, at the very least, employing abductive processes such as numerical heuristics (e.g., the finite difference method) in order to surface properties that are or are not directly knowable due to the incompleteness of the cues presented to learners. Abduction, in an oversimplified sense, pertains to viable inferences in the form of generalizations that can be claimed about classes of objects despite the fact that the information presented to the knower is incomplete.

3. What is abduction?

The notion of abduction is commonplace in disciplinary areas such as linguistics, artificial intelligence, philosophy, and semiotics. Recent investigations in school geometry (Arzarello, Micheletti, Olivero, & Robutti, 1998; Pedemonte, 2001), problem solving (Cifarelli, 1997, 1998, 1999a, 1999b), algebra (Reid, 2003; Rivera & Becker, 2007), and calculus (Ferrando, 2000) have explored the significance of abduction in reasoning, in the formation of conjectures, and in the development of plausible generalizations and proofs. The semiotician Peirce (1958) has pointed out its significant role in the formation of induction. Further, it is not an entirely new concept since it is often employed in daily life and in professional and academic practices such as when doctors perform a medical diagnosis (Charniak & McDermott, 1985), when jurists analyze evidence in court cases (Pennington & Hastie, 1988), or when evolutionary psychologists develop hypotheses on the role of psychological adaptation in human behavior (Holcomb, 1996). In fact, we all make explanatory inferences — that is, abductions — about an object or a phenomenon of interest. Taken together, the trivium of abduction, induction, and deduction provides a more coherent, “complete account of the whole process of inquiry” (Minnameier, 2004, p. 76).

Traditional conceptualizations relevant to the nature of mathematical reasoning uphold the view that deduction and induction form a binary pair in such a way that all non-deductive types of reasoning tend to fall under the category of the other — that is, inductive (Magnani, 2005, p. 267; cf. NCTM, 2000). But Peirce thought otherwise and proposed abduction as a third form of reasoning. A proper understanding of abduction requires carefully delineating between the early and the late works of Peirce (Flach, 1996). Peirce introduced the idea of abduction as early as the 19th century in relation to the epistemological processes of induction and deduction. In this article, we appropriate a later Peircean perspective. Very briefly, the early Peirce framed abduction within the classical, Aristotelian logic of premise-conclusion which he later abandoned in favor a less narrow and more dynamic view that foregrounds inference, including the creative component inherent in abduction (Minnameier, 2004). Further, Peirce conceptualizes induction and deduction as necessary complementary tools to abduction in the sense that they operate at the confirmatory and predictive stage of knowledge construction. Abduction is, thus, seen to be primary in the constructive process. Fig. 2 illustrates the relationship among the three types of reasoning.

Otherwise referred to as discovery reasoning, suggestive reasoning, and explanatory reasoning (Abe, 2003, p. 232), abduction functions primarily in generating and forming hypotheses or theories but not in the naïve sense of “happy guessing” (Hempel, 1966). Formulating hypotheses and theories does not merely involve relying on direct observable facts; it also results from a rigorous “trial and error elimination” method (Hempel, 1966). Josephson and Josephson (1994) have proposed the following form of abduction which basically extends Peirce’s notion of abduction as the generation of hypotheses to include selecting hypotheses that yield the best explanation:

D is a collection of data (facts, observations, givens).
H explains D (would, if true, explain D).
No other hypothesis can explain D as well as H does.

Therefore, H is probably true.

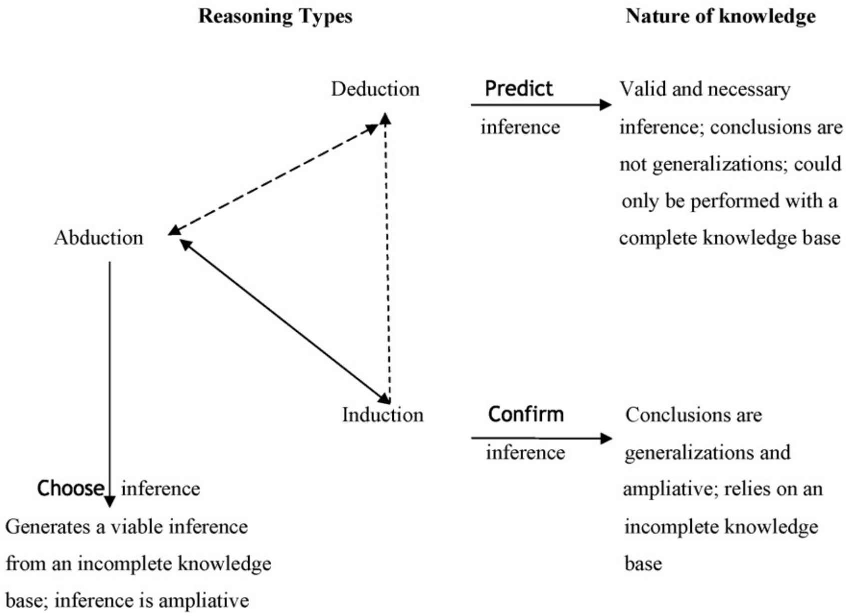

Fig. 2. The trivium of abduction, deduction, and induction.

Abduction and deduction are distinguished in one important aspect. The "truth-preserving" (Josephson & Josephson, 1994, p. 13) nature of deduction always produces a valid and necessary inference or evidence because true premises always yield true conclusions. The inferred conclusions are not generalizations because they have been necessitated by the premises following established rules of deductive inference. Hence, there is no need for empirical verification (Smith, 2002). Abduction, on the other hand, generates fallible or ampliative inferences (Josephson & Josephson, 1994, p. 12). It is "truth-producing" (Josephson & Josephson, 1994, p. 13) that results in the development of ampliative conclusions which contain extra knowledge that has not been or could not be drawn directly from the premises. Thus, some abductive conclusions might be strong, while others tend to be weak. Analogical and causal inferences are two examples of ampliative abductions. Also, Abe (2003) points out that in deduction, "there is no chance for discovery" since it "cannot work with an incomplete knowledge base" (p. 233), while abduction (and induction, for that matter) employs "synthetic reasonings" in the form of discovery "that can deal with incomplete knowledge" (Abe, 2003).

Distinguishing between abduction and induction is a more complicated task. A good starting point in clarifying differences between induction and abduction involves swans and their colors as a way to articulate the problem when an individual claims to make a generalization mainly by relying on appearances. Children in many parts of the world have always believed that the proposition, "Swans are white" is true at all times with the unproblematic belief that all swans in the future will always be white. However, the problem with the proposition has to deal with the fact that it might not be true among a different cohort of individuals. For example, children who live in Australia know swans can be both black and white. What is being pointed out in this situation has in fact more to deal with abduction than induction. That is, the primary issue is how to choose a hypothesis in relation to an observed phenomenon of interest which can then be used to confirm something about the phenomenon. Thus, the problem of choosing and confirming a hypothesis illustrates what Peirce considers to be abduction and induction, respectively. Such a process of transforming possibilities into generalities will always be incomplete, but it always begins with abduction — "the operation which introduces any new idea" (CP 2.264).

For Peirce, induction "does nothing but determine a value and deduction merely evolves the necessary consequences of a pure hypothesis" (CP 5.171). Further, he associates deduction with proving that "something *must* be" (i.e., a logical implication), induction with showing that "something *actually* is operative," and abduction as "merely suggest[ing] that something *may be*" (CP 5.171). Thus, abduction can be seen as being prior to induction and deduction which are viewed as performing a follow-up confirmatory and predictive function, respectively. Abduction fuels the production of (possibly fallible) conjectures leading to the adoption of a testable hypothesis (Abe, 2003, p. 234). Then, induction tests the hypothesis through an experiment which if shown to hold increases its confidence value (Abe, 2003, p. 234).

Abe (2003) writes: "Induction can be formalized as the generalization of examples. [It] finds tendencies in examples and generates general rules (hypotheses) from examples and background knowledge" (p. 234).

The construction of novel concepts, hypotheses, and theories in abduction cannot simply be reduced to the performance of induction because the latter produces knowledge that is "more of the same" (Minnameier, 2004, p. 79). Abduction, however, employs a "creative inferential" process that enables the discovery of new concepts, hypotheses, or theories (Abe, 2003; Magnani, 2005) which "cannot be directly observed" (Abe, 2003, p. 234) and oftentimes "starts from consequences and looks for reasons" (Magnani, 2005, p. 265). While induction has a discovery component, however, what is discovered are oftentimes "tendencies that are not new events" (Abe, 2003, p. 234). Consequently, abduced concepts, hypotheses, or theories basically have the nature of being explanatory inferences that either have a high degree of plausibility (Peirce, 1958) or evolve into an "inference to the best explanation" (following Josephson & Josephson, 1994). Furthermore, like induction, abductive reasoning is oftentimes constrained by the realities of uncertainty and imperfect or incomplete information which influence the strength of plausibility (Josephson & Josephson, 1994, p. 270).

4. Explicating the roles of abduction and induction in generalization

The Dotted Squares Task (Fig. 3) was given to the subjects as a pattern sequence of increasing squares. We were interested in ascertaining how they would abduce a viable formula from the available classes of numerical and figural cues which would then enable them to induce and eventually generalize to any number of arrays. In particular, we address the following questions: How do those subjects who primarily rely on the available figural cues develop their abductive generalizations? In the case of those who rely primarily on the numerical cues, how do they reason about their generalization despite the fact that the figural cues clearly show a perceptually available attribute of each element in the class (i.e., squares)? In the inductive phase, how do the subjects confirm their abductions before stating a generalization?

4.1. Method

4.1.1. Subjects

Forty-two undergraduates (34 women, 8 men) participated in the clinical interview for extra credit. They were enrolled in an introductory course for elementary mathematics teachers in a public university in northern California. Their ages ranged from 19 to 55, with a mean age of 23.42. Racial profile is as follows: 15 Caucasian Americans, 4 African Americans, 11 Asians and Asian Americans, and 12 Hispanic Americans. The task was given to individual participants, and the mean time it took to accomplish the task was 3 min.

4.1.2. Materials

Each triad in the generalizing task was constructed so that the second and third figures and numerical values were related to the first and second figures and numerical values, respectively. The task required all participants to either draw or compute values for two additional cases before they were asked to obtain a generalization.

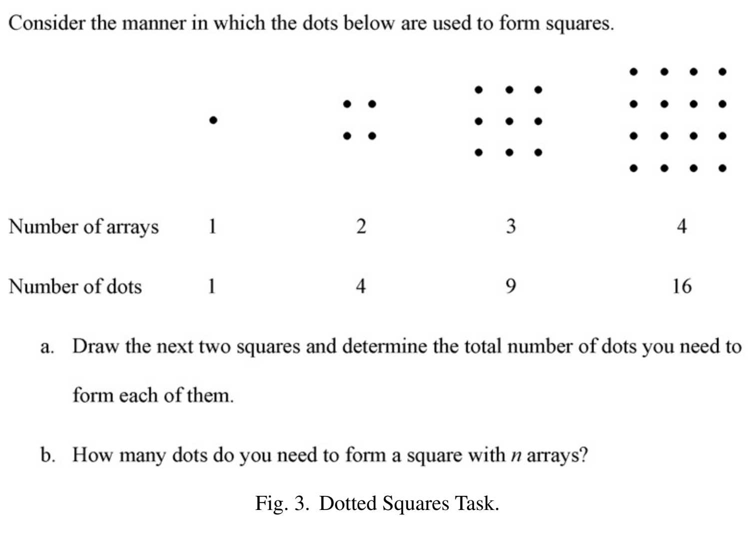

Fig. 3. Dotted Squares Task.

Table 1
Summary of responses for the squares task ($n = 42$)

	Figural	Numerical
• n^2	• 2	• 33
• n		• 1
• $x + 2$ (or $n + 2$)		• 2
• $1 + 3n + 2m$		• 1
• $N \times 32 = 64$		• 1
Did not generalize to a closed form		• 2

4.1.3. Procedure

Subjects were asked to read the task first and then to think aloud while in the process of solving the task. The interview protocol was designed to help them explicitly articulate how they were performing generalization. In particular, the protocol addressed issues such as: (1) whether the numbers they computed and the generalizations they developed were drawn from either the numerical cues or the figural cues; (2) how they justified and confirmed the viability of their generalizations, and; (3) whether there were other ways of establishing and confirming their generalizations.

4.1.4. Results

Table 1 presents a summary of the students' abductive generalizations with at least 83% of the subjects inferring the closed form n^2. Only two participants, MC and JS, primarily relied on the figural cues, while 33 others employed a numerical strategy. MC immediately saw that rows and columns on each figural cue actually contained the same number of dots:

MC: It's n squared.
FDR[2]: So why n squared?
MC: Because I see that from the number of arrays. For 1 there's one dot, 2 there's two columns and two rows and for 3, three columns and three rows. By doing that altogether there's n squared.

JS saw each figural cue to be consisting of multiples of the same row:

JS: So you need n amount of . . . squares. You're using multiples, so it'd be n squared.
FDR: How do you justify that?
JS: For example, remember 2. 2 by 2 so you're on number 2. So you'd put 2 down and there's a row 2 by a row 2. So 2 times 2 is 4. For 3, it'd be 3 times 3. So that's why it's n squared.

In the case of those who primarily relied on the numerical cues, 27 out of the 33 (about 64% overall) quickly recognized the fact that the six-member class $\{1, 4, 9, 16, 25, 36\}$ consisted of "squared numbers" which led them to the closed form n^2 as a possible generalization. The remaining six initially set up a table of values and obtained the consecutive differences in order to generate 25 and 36. They then drew squares consisting of 5 and 6 arrays, counted the dots, and obtained the same numbers. However, in the final stage of generalization, the existing knowledge about the differences was abandoned since they saw a connection between the independent and dependent values (i.e., $1 \to 1^2$, $2 \to 2^2 = 4$, $3 \to 3^2$, etc.). Finally, in confirming the viability of their abductive generalization, all 35 subjects pointed to the given figural cues and assumed that the pattern sequence was increasing.

Several other types of abductions were generated due to the fact that seven subjects perceived relationships among the elements in the same class rather differently. Two of the seven justified the abduction $n + 2$ by claiming that it resulted from the second consecutive finite difference, as shown in Fig. 4. For example, SM obtained the correct number of dots for arrays 5 and 6 by drawing the dotted squares and then counting the dots in each array one by one. She then shifted to a numerical strategy when she computed the first consecutive difference between two successive numerical cues and obtained the numbers 3, 5, 7, 9, and 11. Even if she recognized that with five dots, a

[2] FDR refers to the first author who conducted the interview with all 42 participants.

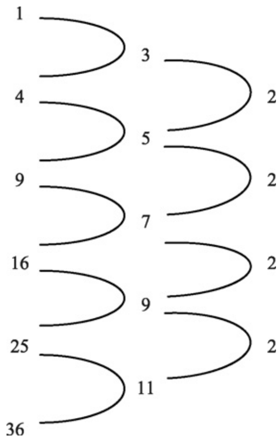

Fig. 4. Explanation for $n+2$.

square had to have length 5, she wanted to establish a generalization on the basis of the second consecutive common difference:

12	SM:	How many dots do you need to form a square with *n* arrays? This one's
13		more complicated because . . . This must be *n* plus . . . Hmm, this increases
14		by 2. $n+2$, I guess.
15	FDR:	So how do you justify the *n* plus 2? What does that mean?
16	SM:	*n* plus 2 . . . *n* represents the length and the width. I don't know. Ahm. I've
17		a problem with it.

Two students, CS and RT, produced two different abductions as follows. In thinking about the abductive generalization *n*, CS thought that because a square with *n* arrays could be formed with *n* dots in each array, then the general expression *n* was sufficient:

18		You need *n* dots because with n dots you can form an array which you use to
19		form a square . . . like . . . with 2 dots I can form two arrays with 2 dots each, 3
20		dots here form 3 arrays with 3 dots in each. That's how I formed these two
21		squares [referring to squares with 5 and 6 arrays] . . . It should be *n*, I think.

RT thought about her abductive generalization, $n \times 32 = 64$, in the following manner:

22	RT:	It's been a long time since I've done geometry. So I don't remember
23		formulas very well.
24	FDR:	That's okay. Just think about it.
25	RT:	[Reads the problem.] Draw the next two squares. Okay, I get that. Okay,
26		so I will need 2, 3, 4, 5 dots. Two squares. So that would be double its
27		size. Then I would need what's 16 plus 16? Ahm, hold on. Times 2 is 32.
28		So you'll need 32 dots.
29	FDR:	What about the next one?
30	RT:	How many dots do you need to form a square with n arrays? Ah . . . not
31		sure. So what would I do. I don't know if I would go . . . oh let's see, 32. I
32		don't wanna do plus 'coz I've done too many pluses so I've to think for a
33		minute . . . 32 times 2 is ah 64. So $n \times 32$ is . . . like this doubling in size,
34		so it's 64.

For RT, the numbers 1, 4, 9, and 16 resulted in a process of doubling instead of squaring. Hence, the next two extensions she obtained according to her formula of doubling were 32 and 64. Further, since task B required her to develop a generalization for a square with *n* arrays, she produced a "recursive type" formula by multiplying 32 by *n* which she then equated to 64 (similar to the form Current $\times$ n = Next).

Two students, YH and MG, were unable to state a closed formula but nonetheless abduced in the following manner. Initially, YH quickly obtained the total number of dots that was required to form squares with five and six arrays since she recognized them to be perfect squares. However, in developing a generalization, she wanted to develop a formula that would take into account the two successive differences she obtained from the extensions (see Fig. 4). Unfortunately, she was unable to find one. In the case of MG, she was unable to state a generalization because she obtained incorrect values for the two additional cases which made it difficult for her to recognize common features. She initially obtained the first consecutive differences from the given terms $\{1, 4, 9, 16\}$. However, in forming squares with five and six arrays, she merely extended the given square with four arrays by adding one and two columns of dots on the right side of the square, respectively, in which case she then counted 20 and 24 dots. She then could not make sense of the class $\{1, 4, 9, 16, 20, 24\}$.

4.1.5. Discussion

The two students who abduced the generalization n^2 from the figural cues were significantly influenced by their perception about the way in which the square arrays were formed. In fact, they quickly saw that the corresponding numerical cues resembled perfect squares. What was relevant in their abduction process was their cognitive capacity to draw on their stored perceptual knowledge — that is, their structured information about the dimensional property of squares — perhaps as a consequence of previous mathematical experiences that enabled them to form a viable explicit, variable-based representation. While they immediately saw a square among the figural cues, they used a property of a square in abducing and in explicitly surfacing the form n^2. Magnani (2005) refers to this type of abduction as *model-based* whereby the students used algebraic symbols to articulate an important property about the sequence of figural cues, or in "transforming knowledge from its tacit to its explicit form" (Magnani, 2004, p. 517). Another manifestation of model-based abduction involves numerically recognizing the basic structure of squared numbers. Without relying on the figural cues, some participants decomposed 1 into 1^2, 4 into 2^2, 9 into 3^2, and so on which also enabled them to abduce the form n^2. As a matter of fact, they referred to the figural cues only when they were prompted to confirm their generalization in another way.

In performing the inductive task of confirming an abduced generalization, the subjects basically used the available and constructed figural cues to show that the general form n^2 referred to the fact that any square array had either an equal number of rows and columns or multiples of the same row. Both abductive and inductive reasoning complemented each other in the sense that while it was easy for the subjects to generate and justify a possible abductive generalization, it was only in the inductive phase whereby they tested the plausibility of the abduced form by examining extensions (i.e., particular cases beyond what has been made available for them). While those who came up with the general forms n and $n+2$ thought their justifications in the abductive phase were reasonable, they eventually abandoned the forms in the inductive phase due to the relative insufficiency of either form in fully capturing in symbolic terms a general attribute that would yield the total number of dots for any given far generalization task (such as how many dots all in all in a square with 127 arrays).

5. Figural and numerical approaches to establishing generalizations in the abductive phase

We gave the Squares Task and the Hexagons Task (Fig. 5) to the participants in order to fully assess their different abilities in abducing rules of relationship or invariant properties within classes of cues that have been represented in both figural and numerical form. Considering the lack of further specific assumptions and explicit contexts in which to think about the pattern tasks, participants were allowed to produce various rules or properties which they could express using several different algebraic forms involving one or two variables and with the forms possibly taking the shape of a recursion, a direct expression, or some other viable means. We note that in the elementary and middle school mathematics curricula, the two tasks are typical generalizing problems that teachers oftentimes use to illustrate classes that are arithmetical sequences in the linear order. Hence, there is a tacit assumption that each class is an increasing sequence with the polygons continuing to grow along one dimension only. The participants were asked to generate two additional cases from the known cases on each task. Then they were asked to abduce a generalization in whatever form that made sense to them. The same procedure protocol (Section 4.1.3) was used on both tasks.

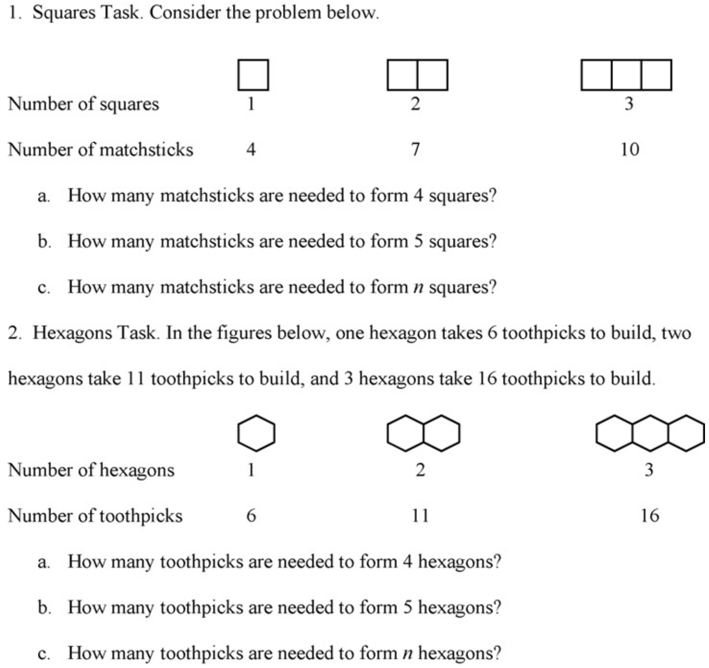

Fig. 5. Squares task and hexagons task in compressed form.

5.1. Results

Tables 2 and 3 present the frequency of responses and the response patterns of participants who used either figural or numerical cues in abducing a generalization. On both tasks, they saw the numerical cues as providing better support than the available figural cues in developing an abduction (with a mean of 65%). The most frequent abductive process that used the numerical cues involves the method of *finite difference* (see Fig. 8 for an illustration). On average, 39% claimed that the relation among the dependent values on the first and the second task could be expressed by the formulas $n+3$ and $n+5$, respectively. When prompted to justify the viability of the generalizations, they pointed to the common differences which they obtained from the numerical cues and did not link the differences back to the figural cues. Also, due to lack of notational fluency, the variable n in $n+3$ and $n+5$ was defined to mean "the number of matchsticks (or toothpicks) before it."

In the case of LM who abduced the form $4+3n$ from the available numerical cues, she primarily *relied on the sequence of dependent terms* $\{4, 7, 10, 13, 16\}$ without taking into account how the values were related to the squares being formed. Thus, she assumed that the variable n would implicitly take on values beginning with 0. She claimed a similar argument in the case of the abduced form $6+5n$. But when she was asked to explain what the formulas meant in the context of the task and, in particular, the specific case when $n=0$, LM was unable to justify them.

Table 2
Summary of responses for the squares task 1 ($n=42$)

	Figural	Numerical
• $3n+1$	• 5	• 5
• $4+(n-1)3$	• 1	
• $4+3n$	• 4	• 1
• n (or x) $+3$	• 5	• 16
• $4n$		• 1
• $n+(n-1)=\#n-1$		• 1
• $n\times 4-1$	• 1	
• $n+3=16$		• 1
Unable to generalize		• 1

Table 3
Summary of responses for the hexagons task 2 ($n = 42$)

	Figural	Numerical
• $5n + 1$	• 4	• 6
• $6n - n + 1$		• 1
• $6 + (n - 1)5$	• 1	
• $6 + 5n$	• 4	• 1
• n (or x) + 5	• 5	• 17
• $n + (n-1) = \#n - 1$		• 1
• $26 + n = 31$		• 1
Unable to generalize		• 1

Five participants used *guess and check* in abducing the general forms $3n + 1$, $5n + 1$, and $6n - n + 1$. In the case of the form $3n + 1$, they initially constructed a two-column table showing number of squares in the first column and number of matchsticks in the second column. Then they obtained the common difference between two consecutive rows in the second column, wrote down $3n$, and claimed that each dependent term was "always 1 more than 3 times n." A similar process was claimed in the case of the form $5n + 1$.

GL employed a painstaking numerical process of *trial and error* in abducing the form $3n + 1$ (refer to Fig. 9 for his written solution). He began with the form $4n - 1$ and computed the value for $n = 1$. Because the value he obtained was 3, he then tried $4n - n$ and evaluated this expression for $n = 1$. Seeing that he needed 1 more to obtain the first term, 4, he added 1 to $4n - n$. Once again he evaluated $4n - n + 1$ and saw that it produced the correct dependent values in cases $n = 2$ and 3. When he was asked to explain the viability of the abduced form $4n - n + 1$, he reasoned as follows:

FDR: I understand where the $4n$ comes from. But this — $n + 1$ is not clear to me. Tell me more.

GL: Okay. Well I see here that four matchsticks equals one square. So it will just keep doubling down to 8 but it doesn't show us here. It's not. 1, 2, 3, 4, 5, 6, 7 coz you're using that 1. So that's 2 [*referring to the two squares*]. So what I've tried to do is just go through like a shortcut and cheat coz I want 4, 8 but I know I just had to take 1 away to get 7. So now I jumped ahead to 12. Okay but I know it's not because I have to subtract maybe one or two. So 1, 2, 3, 4, 7, 8, 9, 10. So here I subtracted 1 [*referring to the first case*], here I subtracted 2 [*referring to the second case*]. Now with three squares, I have to subtract 3 from what normally would make up one square. So it will be 16 subtracted by 3, 13 and I'll try that out. Okay and here [*referring to the third* case] I have 10, 11, 12, 13. So what I didn't know how to do was how to keep saying if I keep adding on squares I have to subtract that many. So it's like taking away 1 from 8, 2 from 12, 3 from 16 . . . so that's why I have this [*referring to 4n − n+1*]

From the above transcript, the form $4n - n + 1$ actually involves two subgeneralizations, that is, $4n$ and $n + 1$. He knew that each square required four matchsticks which would explain the term $4n$. But he also knew that he had to subtract 1, 2, 3, . . ., $(n + 1)$ matchsticks in each case beginning with the second case. However, GL was not aware that the forms $4n - (n + 1)$ and $4n - n + 1$ were not equivalent. In fact, he then simplified the former expression and obtained $3n + 1$. Finally, he verified that the numbers 4, 10, 13, and 16 could actually be drawn from the form. When GL was asked if he could abduce in another way, he claimed the following:

Okay it has to be a minimum of 4 matchsticks to be n squares. So . . . so it's a minimum of 4 . . . 4 matchsticks . . . but it could go on forever. But you need something like what we were doing. . . . So hold on. [*He writes 4 + 3n.*] I just wrote that to try it out. 4 matchsticks plus 3 times n. Okay now I'm starting to think minus 1 since as I keep adding squares on I have to keep subtracting. . . . I can see maybe 4 matchsticks but I'm multiplying. See I don't know what the 3s are there for.

GL used the same trial-and-error abductive strategy in figuring out the form $6n - n + 1$ for the hexagons task.

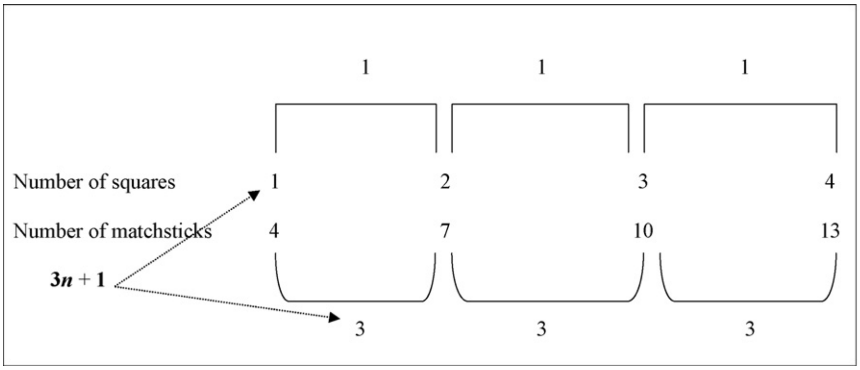

Fig. 6. KM's numerical method in abducing the form $3n+1$ for the squares task.

KM employed a different numerical strategy in obtaining the forms 3n + 1 and 5n + 1. First, she used the given table of numbers and column by column computed the difference between the independent and dependent values in the same column (Fig. 6). Because a pattern emerged from the differences that she computed for the first three cases, she then extended the table by simply following the pattern. This enabled her to determine the number of sticks for the next two cases. However, in establishing the abduced forms, she primarily relied on the first difference, 3, and claimed that adding 1 (pointing to the first term in the first row of values) to 3n would suffice in generating all the dependent terms of the sequence.

In sum, the 22–25 participants who abduced primarily through the numerical cues used several numerical strategies such as finding a common difference, guess and check, and trial and error. However, even if they felt confident that the general forms they generated were sufficiently viable because they were in fact true in all the cases they tested in the follow-up inductive phase of generalization, they still could not justify their viability other than the fact that they worked for the numerical values used. Another issue which they had difficulty addressing was whether the abduced forms they inferred were sufficient enough to describe what they perceived to be the features of the classes they were generalizing (based on a no-response to questions such as "Is that the only possible formula?" or "Can the numbers be generated by using a different expression?").

In the case of the five to six subjects who abduced a generalization primarily on the figural cues, they immediately saw a relationship among the drawn cues. In fact, the general forms they abduced reflected the manner in which they interpreted the drawn figures, including the ones they were asked to construct. Also, the abduced generalizations signified a constructive process which for them remained uniform and invariant. For example, in abducing the form $3n+1$, JS began by computing the common difference, 3, and then pointed out that 3 was the number that would determine the "difference between one figure to the next" since forming a new square in a succeeding pattern meant adding three new matchsticks:

You're trying to make ahm a full square with four matchsticks and if you
already have one side then you would be adding three more on to it depending on
the number of squares that you wanna make 'coz that's how many you're gonna
put, that's how many threes you're gonna add on.

Furthermore, those subjects who abduced from the figural cues employed symbols as a vehicle for expressly articulating a generalized relationship. For instance, in the transcript below, CG explained how he abduced the form $4+(n-1)3$ for the squares task:

How many matchsticks are needed to form four squares? So ahm I'm
looking for a pattern. For every square you add three more. So let's see. So
that would be four plus three for two squares. Plus three more would be for three
squares. So it's ten matchsticks. So you have four. So there would be thirteen. So
thirteen plus three more is sixteen . . . So for three squares, it would just be two
threes. So there'd be two threes, three threes is for four squares, and four threes
for five squares. For n squares, it would just be ahm n minus 1 threes.

5.2. Discussion

Participants who abduced numerically developed formulas that could be characterized as more practical and situational than conceptual. Further, they justified the viability of the forms they produced through a mere appearance match. For example, they verified in the induction phase that the abduced forms $4+3n$ and $6+5n$ correctly generated the two classes beginning with the case when $n=0$ without considering what the variable n meant in relation to the task being analyzed (based on a no-response to the question "What does $n=0$ mean with respect to the problem?"). The general forms they constructed by way of a numerical strategy could be viewed in Magnani's (2005) terms as having the character of a *manipulative abduction* in the sense that the forms were primarily developed on the basis of having been drawn from either executing an efficacious or automatized mathematical procedure or using a practical approach in surfacing features and possible fit without the benefit of a conceptual explanation (cf. Magnani, 2004). For Magnani (2005), a manipulative abduction develops in "situations where we are thinking through doing and not only, in a pragmatic sense, about doing" (p. 274).

Further, while the reasoning and the representational forms that the participants produced involve communicable knowledge, they, however, did not oftentimes reach the status of a conceptual explanation whose viability could be rigorously evaluated. For example, those who abduced numerically could easily use the finite difference method automatically as a consequence of repeated engagement with the method. Unfortunately, in most cases they did not provide a conceptual basis in justifying the viability of the abduced forms they produced because they perceived the numerical method as an operation that had the characteristic of being practical and situational. Hence, it is noteworthy to raise the issue of *instability of the abductive process* with respect to subjects who employed numerical strategies since many of their abductive processes were task-specific in intent — that is, they worked but only within the context of the operations they used to accomplish the objectives in a task. For example, GL and KM's abductive processes would not work if they were used in a different situation.

Participants who abduced figurally saw relationships within classes of figural cues. Their abductive processes can be summarily described as performing a kind of relational similarity within classes in which the focus was *not* on the individual cues in a class *per se* but on a possible invariant relational structure that was perceived to exist between and, thus, projected onto the cues. Further, since they immediately saw an invariant relational structure among the figural cues, the corresponding numerical cues did not matter significantly in their abduction.

Variables also played an important role in expressly articulating an abduction. Participants who abduced figurally introduced variables in relation to their need to capture in symbolic terms an invariant action on the figural cues. For example, the abductive form of CG (Fig. 7) reflects the use of a variable which substitutes as a general placeholder or expression for a sequence that he perceived to continuously grow indefinitely. Participants who abduced numerically employed variables rather inconsistently. In fact, some were confused with what the variables meant at various stages in their abduction, that is, they could not figure out whether the variable n pertained to the number of squares (or hexagons) formed or to the number of matchsticks (or toothpicks) needed to form squares. The written solutions of RM and GL (Figs. 8 and 9) illustrate how variables were loaded with no other algebraic intent except in relation to constructing a general expression that would best produce the dependent terms upon substitution of a value in the relevant domain.

6. General discussion and provisional closure

Abduction and induction play an important role in generalizing knowledge from a finite, incomplete class of particular instances (whether figural, numerical, or everyday objects). A valid and viable generalization reduces the amount of time and energy needed if individual instances out of a possibly infinite number of cases were to be investigated one at a time. Researchers have underscored the need for all learners to develop meaningful abductive processes leading to the construction of general knowledge since it is intimately connected to larger issues involving mathematical abstraction (Cifarelli, 1999b; Davydov, 1990; Dreyfus, 1991; Harel & Tall, 1989; Tall, 1991).

In this article, we described how two different representational cues (figural, numerical) assisted the participants in abducing general forms from incomplete classes of cues, including how they confirmed the viability of the forms through induction that eventually led to an established generalization. Fig. 10 has been derived from Fig. 2 with additional information drawn from the findings in the study. Given the classes of cues that the participants had to deal with in all three generalizing tasks (Figs. 3 and 5), a preference for numerical or figural abductive strategies seems to

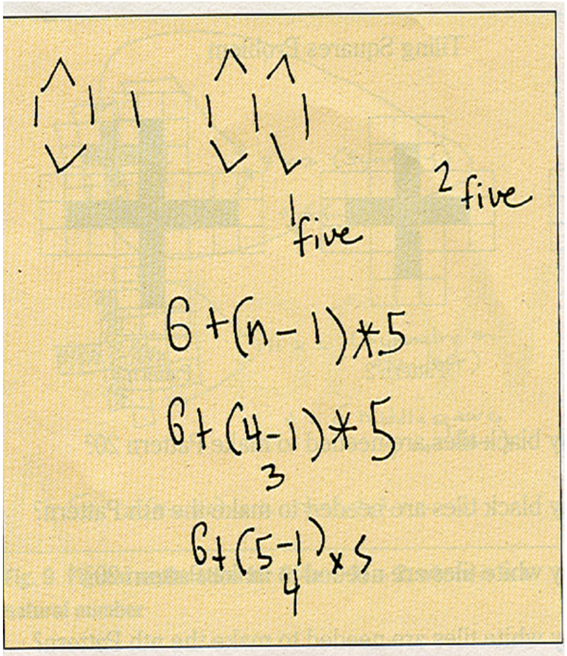

Fig. 7. Written solution of CG on the hexagons task.

affect in significant ways the manner in which abduced forms are discovered, constructed, and generalized in symbolic terms. Further, at the inductive phase, testing and confirming the viability of an abduced form, while definitely not deductively valid processes, assist significantly in deciding whether the stated premises (i.e., the proposed general form as a rule, the available cues as cases) make it reasonable to accept the conclusion (i.e., the generalization). We also note that while the participants who abduced numerically developed more strategies than those who abduced figurally, the latter group was more capable than the former in justifying the viability of, including selecting the best, abduced forms.

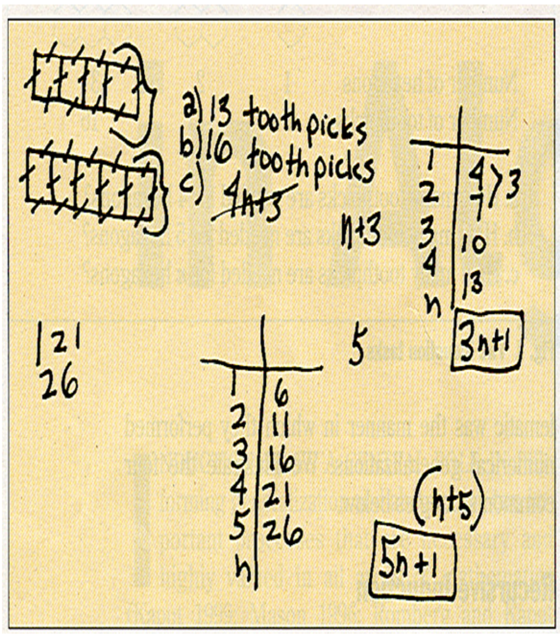

Fig. 8. Written solution of RM on the squares task and the hexagons task.

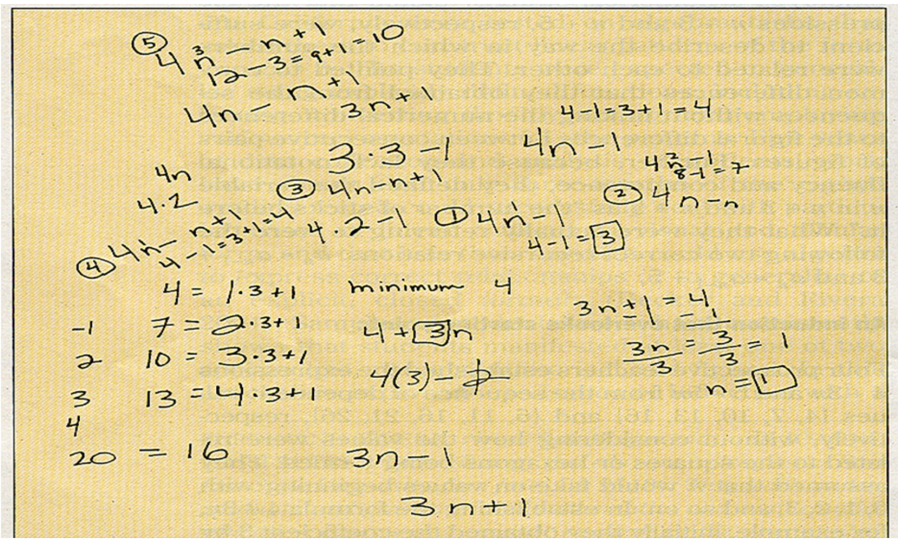

Fig. 9. Written solution of GL on the squares task.

Another interesting observation about the relationship between abduction and induction in generalization involves what we refer to as the *progressive formalization of forms*, a view very similar to the ones suggested by Aliseda (1996) and Abe (2003). In the abductive phase, the participants would oftentimes discover and construct a viable form *R* for a class *not* on the basis of the available elements or the entire class itself but only on one or two cases (cues). (See, for example, the abductive reasonings offered by MC and JS in Section 4.) Then, in the inductive phase, when the participants have repeatedly tested and confirmed the viability of *R* in several more cues, *R* is further generalized into the form *F* which would then be assumed to hold for the entire class (Aliseda, 1996). (See, for example, the abductive–inductive reasoning of CG in Section 5.) In a similar vein, the inductive form *F* could also be viewed as having resulted in "tendencies that are not new events" (Abe, 2003, p. 234).

Our final two points concern the issue of possibly having different types and forms of abductions for a given class of abstract objects. *First*, concerning the issue of types of abduction, in Sections 4 and 5, we extrapolated model-based

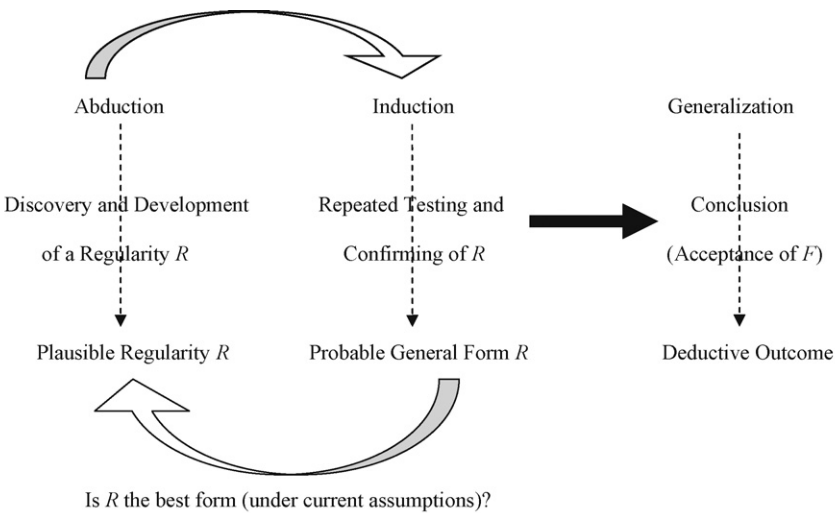

Fig. 10. The abductive–inductive reasoning process on linear patterns.

and manipulative abductions which are classified as extra-theoretical abductions.[3] We assume there are still many other forms (see, e.g., Thagard (1988)) that could (better) explain the process of generalization of pattern sequences of oftentimes incomplete classes of abstract objects. Further, we explored abduction as a way of complicating the conversation about how we conceptualize the process of generalization in mathematics which oftentimes tends to be equated with induction alone (Dreyfus, 1991; Harel & Tall, 1989). Merely associating generalization with the act of inducing from particulars or performing iterated actions seems to ignore the creative component that Peirce (1935) has pointed out as being embedded in abduction. Abduction for us provides a more robust characterization of how individuals begin to actually construct and establish generalizations. Generalization is, thus, seen as an outcome of a combined abduction–induction process.

Second, concerning the types of abduction that are possible with patterning tasks in algebra, we surface the problematic and prevalent practice in which such tasks are oftentimes stated in textbooks with a seemingly taken-as-shared assumption that the sequences were to evolve in a particular manner across contexts of any kind and, thus, are expected to take the shape of a predictably correct general form. For example, all the participants in the study interpreted the two patterns in Fig. 5 as linear sequences. But a more rigorous mathematical analysis of the two patterning tasks reveals the ambiguous nature inherent in such tasks — that is, it is possible to produce equally viable and valid abductions that could be inductively supported under different sets of assumptions. However, such tasks run the risk of becoming algebraically insignificant, that is, incapable of being expressed or too complex to be written down in some defined closed form (cf. Lee (1996)). Further, while abductive inferences about patterns may open up the possibility of having several different and correct generalizations, they also have to be "good abductions" in the same manner that Kyrburg (2001) in the introduction above asserted about the prescriptive element inherent in logic. Good abductively drawn and inductively tested generalizations involving patterns in algebra should be able to explain why they hold considering the givens and the unknowns and be tested on the basis of some rigorous experimental verification. *Pace* Peirce:

> What is good abduction? What should an explanatory hypothesis be to be worthy to rank as a hypothesis? Of course, it must explain the facts. But what other conditions ought it to fulfill to be good? The question of the goodness of anything is whether that thing fulfills its end. What, then, is the end of an explanatory hypothesis? Its end is, through subjection to the test of experiment, to lead to the avoidance of all surprise and to the establishment of a habit of positive expectation that shall not be disappointed. Any hypothesis, therefore, may be admissible, in the absence of any special reasons to the contrary, provided it be capable of experimental verification, and only insofar as it is capable of such verification. (CP 5.197)

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References

Abe, A. (2003). Abduction and analogy in chance discovery. In Y. Ohsawa & P. McBurney (Eds.), *Chance discovery* (pp. 231–248). New York: Springer-Verlag.

Aliseda, A. (1996). A unified framework for abductive and inductive reasoning in philosophy and AI. In P. Flach & A. Kakas (Eds.), *Contributing paper to the ECAI '96 workshop on abductive and inductive reasoning*. Budapest, Hungary.

Arzarello, F., Micheletti, C., Olivero, F., & Robutti, O. (1998). A model for analyzing the transition to formal proofs in geometry. In A. Olivier & K. Newstead, *Proceedings of the 22nd conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 24–31). Stellenbosch, South Africa.

[3] Magnani (2005) defines *theoretical or selective abduction* as a philosophical-based abduction that is best exemplified in computational programs or in particular types of diagnostic reasoning. For example, if we see smoke in the forest, we might initially surmise that there was a fire at some point prior to the appearance of the smoke. Thus, the inference made (i.e., "there was a fire") was not a result of deduction but was an explanatory plausible inference. Theoretical abductions are classified as selective abductions in the sense that the hypotheses generated can still be subject to a further evaluation, and that such an evaluation involves assessing for strength of plausibility, with abductions yielding the best explanation as being the most plausible. A follow-up deductive process involves exploring logical consequences, while a follow-up inductive process involves testing the inference with available, relevant data.

Chandrasekaran, B. (1997). Diagrammatic representation and reasoning: Some distinctions. *Invited paper for the association for the advancement of artificial intelligence symposium series on diagrammatic reasoning.* Boston, MA. Available online: www.cse.ohio-state.edu/~chandra/dr-II.pdf

Charniak, E., & McDermott, D. (1985). *Introduction to artificial intelligence*. Reading, MA: Addison-Wesley.

Cifarelli, V. (1997). Emergence of abductive reasoning in mathematical problem solving. Paper presented to the Special Interest Group for Research in Mathematics Education. *Annual Conference of the American Educational Research Association.* Chicago, Illinois.

Cifarelli, V. (1998). Abductive actions in mathematical problem solving: Connections between problem solving and problem posing. Paper presented to the Special Interest Group for Research in Mathematics Education. *Annual Conference of the American Educational Research Association.* San Diego, California.

Cifarelli, V. (1999a). Abductive inference: Connections between problem posing and solving. In O. Zaslvasky (Ed.), *Proceedings of the 23rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 217–224). Haifa, Israel: The Technion Printing Center.

Cifarelli, V. (1999b). Abduction, generalization, and abstraction in mathematical problem solving. In C. Spinks & J. Deely (Eds.), *Semiotics 1998: Proceedings of the 23rd annual meeting of the semiotic society of America* (pp. 97–113). Toronto, Canada: Peter Lang.

Davydov, V. (1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordrecht, Netherlands: Kluwer.

Ferrando, E. (2000). The relevance of Peircean theory of abduction to the development of students' conceptions of proof (with particular attention to proof in calculus). In C. W. Spinks & S. Simpkins (Eds.), *Proceedings of the twenty-fourth annual meeting of the Semiotics Society of America: Semiotics 2000* (pp. 1–16). New York: Legas.

Flach, P. (1996). Abduction and induction: Syllogistic and inferential perspectives. In P. Flach & A. Kakas (Eds.), *Contributing paper to the ECAI '96 workshop on abductive and inductive reasoning*. Budapest, Hungary.

Gelman, S., & Wellman, H. (1991). Insides and essences: Early understandings of the non-obvious. *Cognition*, *38*, 213–244.

Harel, G., & Tall, D. (1989). The general, the abstract and the generic in advanced mathematical thinking. *For the Learning of Mathematics*, 38–42.

Hempel, C. (1966). *Philosophy of natural science*. Englewood Cliffs, NJ: Prentice-Hall.

Holcomb, H., III. (1996). Just so stories and inference to the best explanation in evolutionary psychology. *Minds and Machines*, *6*, 525–540.

Holland, J., Holyoak, K., Nisbett, R., & Thagard, P. (1986). *Induction: Processes of inference, learning, and discovery*. Cambridge, MA: MIT Press.

Josephson, J., & Josephson, S. (1994). *Abductive inference: Computation, philosophy, technology*. New York: Cambridge University Press.

Kyrburg, H. (2001). Real logic is nonmonotonic. *Minds and Machines*, *11*, 577–595.

Lee, L. (1996). An initiation into algebra culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 87–106). Dordrecht, Netherlands: Kluwer Academic Publishers.

Magnani, L. (2004). Conjectures and manipulations: Computational modeling and the extra-theoretical dimension of scientific discovery. *Minds and Machines*, *14*, 507–537.

Magnani, L. (2005). An abductive theory of scientific reasoning. *Semiotica*, *153*(1–4), 261–286.

Mervis, C., Johnson, K., & Scott, P. (1993). Perceptual knowledge, conceptual knowledge, and expertise: Comment on Jones and Smith. *Cognitive Development*, *8*(2), 149–155.

Minnameier, G. (2004). Peirce-suit of truth: Why inference to the best explanation and abduction ought not to be confused. *Erkenntnis*, *60*, 75–105.

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics.* Reston, VA: Author.

Pedemonte, B. (2001). Some cognitive aspects of the relationship between argumentation and proof in mathematics. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 33–40). Utrecht, New Zealand.

Peirce, C. S. (1958). *Collected papers of Charles Sanders Peirce* (Vols. 1–6). In C. Hartshorne & P. Weiss (Eds.). Cambridge, MA: Harvard University Press. [Referenced in the text as follows: CP *X.abc* refers to Vol. *X*, paragraph *abc*.].

Pennington, N., & Hastie, R. (1988). Explanation-based decision making: Effects of memory structure on judgment. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *14*(3), 521–533.

Reid, D. (2003). Forms and uses of abduction. Paper presented to Working Group 4: Proof and argumentation. *Third Annual Conference of the European Society for Research in Mathematics Education*. Bellaria, Italy. Available online at: http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG4/TG4_Reid_cerme3.pdf.

Rivera, F. & Becker, J. (2007). Abduction in pattern generalization. In W. Jeong-Ho, L. Hee-Chan, P. Kyo-Sik, & J. Dong-Yeop (Eds.), *Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 97–104). Seoul, Korea: Korea Society of Educational Studies in Mathematics.

Shipley, E. (1993). Categories, hierarchies, and induction. In D. Medin (Ed.), *The psychology of learning and motivation* (pp. 265–301). San Diego, CA: Academic Press.

Smith, L. (2002). *Reasoning by mathematical induction in children's arithmetic*. Oxford, UK: Pergamon.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 3–24). Dordrecht, Netherlands: Kluwer.

Thagard, P. (1988). *Computational philosophy of science*. Cambridge, MA: MIT Press.