Teacher to Teacher

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Figural and Numerical Modes of Generalizing in Algebra

INDUCTION PLAYS A CENTRAL ROLE IN PERforming generalization and abstraction, two important processes that are necessary and highly valued in all areas of mathematics (Kaput 1999; Mason 1996; Romberg and Kaput 1999; Schoenfeld and Arcavi 1988). From 2000 to 2004, at least 30,000 eighth-grade students in northern California were tested on algebra tasks that asked them to construct linear patterns of the form $y = mx + b$. The students were expected to generalize using explicitly defined functions, including selecting, converting flexibly among, and employing various representations for, the patterns. Five years of data collection and analysis of students' work have shown that only three-fourths of the eighth graders tested could successfully deal with particular cases of linear patterns in visual and tabular form, and that less than one-fifth could use algebra to express correct relationships or to generalize to an explicit, closed formula (Becker and Rivera 2004). Samples of students' work have consistently shown that students manifested at least one of two approaches, namely, numerical and figural. In this article, we explore these issues of induction with a different, but still relevant, set of participants: prospective elementary and middle school teachers who took part in our investigation.

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When individuals look at sequences of numbers that possess implicit rules for generating them, they are likely to have different perceptions about the relationships among the numbers. Howard Gardner's (1993) theory of multiple intelligences tells us that some individuals can recognize such relationships spatially or visually, whereas others can detect them logically and mathematically. Raymond Duval's (1998) theory that learners visualize objects and relationships in geometry either perceptually as mere objects or discursively as possessing properties also makes sense when recognizing patterns in algebra. That is, some students may perceptually see arithmetical sequences of numbers as mere numbers that have no connections among each other, whereas others may discursively see relationships that generate the numbers. The main pedagogical point of both theories is that in asking students to perform induction, teachers need to take into account the possible differences in students' mathematical thinking and visualizing processes.

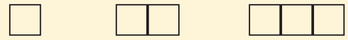
We interviewed forty-two undergraduate elementary and middle school majors to find out how they performed inductive reasoning on two algebra tasks (see **fig. 1**) that involved arithmetical sequences of numbers and figures. Each task contains a sequence of figural and numerical cues. The numerical cues follow a certain arithmetical order. We use the term *figural* to mean that the pictures shown are more than drawings; they also possess attributes or exhibit relationships among one another.

We believe that a middle school algebra curriculum can be made interesting for students if both conceptual and computational tasks can be explained in geometric, visual terms (Driscoll 1999; NCTM 2001). Prior experiences and learnings from the history of algebra may have given middle school teachers the impression that obtaining a generalization is a simple procedural matter that involves using variables and other numerical operations. However, this need not be the case nor the only choice for students. Children and young adults have been known to possess a strong intuitive, visual grasp of mathematical ideas and concepts. Hence, it might be more advantageous if algebra instruction at the middle school level were to capitalize on what young learners can initially accomplish so that they achieve success and can meaningfully progress mathematically to more formal and abstract approaches and models. Thus, we claim that students' ability to reason on induction tasks should *not merely* be about establishing a formula for a pattern by following some rule or technique (such as the widely popular method of *finite differences*). Reasoning should also involve convincing oneself of the validity of the formulas that he or she generates by using a variety of numerical and figural methods. A numerical mode of inductive reasoning uses algebraic concepts and operations (such as finite differences), whereas a figural mode relies on relationships that could be drawn visually from a given set of particular instances. Further, a figural approach could be shown to be as rigorous and analytic as a numerical approach. In inductive reasoning, students should be able to explain how they arrive at formulas and patterns *and* why they make sense.

Generalizing Numerically

TWENTY-SIX OF THE FORTY-TWO PROSPECTIVE teachers we interviewed were predominantly more numerical than figural when they were asked to perform induction on the two tasks given in **figure 1**. They developed generalizations from among the already known and computed numerical values, and they paid little or no attention to the accompanying figural cues. We were not surprised by this result. Oftentimes, prior mathematical experiences required students to obtain formulas from sequences of numbers using algebraic methods such as finite differences regardless of what they might possibly mean in figural terms. What we found rather problematic was the manner in which they performed numerical generalizations. We illustrate the four common strategies below.

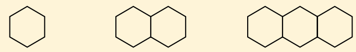
1. Consider the problems below.



Number of squares	1	2	3
Number of toothpicks	4	7	10

a. How many toothpicks are needed for 4 squares?
b. How many toothpicks are needed for 5 squares?
c. How many toothpicks are needed for n squares?

2. In the figures below, 1 hexagon takes 6 toothpicks to build, 2 hexagons take 11 toothpicks to build, and 3 hexagons take 16 toothpicks to build.



Number of hexagons	1	2	3
Number of toothpicks	6	11	16

a. How many toothpicks are needed for 4 hexagons?
b. How many toothpicks are needed for 5 hexagons?
c. How many toothpicks are needed for n hexagons?

Fig. 1 Two induction tasks

Recursive Induction

FOR TASKS 1 AND 2, SEVENTEEN OF THOSE WHO employed a numerical strategy stated that the expressions $n + 3$ and $n + 5$, respectively, were sufficient to describe the way in which the numbers were related to each other. They pointed to common differences that they obtained from the sequences without linking the numerical differences to the figural differences between consecutive pairs of figures. Further, because they lack notational fluency and competence, they defined the variable n in $n + 3$ and $n + 5$ as "the number of sticks before it." What they were actually referring to were the following two correct recursive relations: $a_n = a_{n-1} + 3$ and $a_n = a_{n-1} + 5$.

An induction that overlooks starting points

Four prospective teachers established the expressions $4 + 3n$ and $6 + 5n$ from the sequence of dependent values {4, 7, 10, 13, 16} and {6, 11, 16, 21, 26}, respectively, without considering how the values were related to the squares or hexagons being formed. They assumed that n would take on values beginning with 0, 1, 2, 3, and so on. In establishing the formula $4 + 3n$, for example, initially they obtained the coefficient 3 by taking differences of several consecutive pairs of terms. Then they added the first term, 4, to multiples of the common difference. A similar process was made in the case of $6 + 5n$. When asked to justify why they thought the formulas they obtained were correct, they pointed to the numerical sequences without referring them back to the figural cues.

Finite differences

A rather successful numerical strategy involves the use of *finite differences*. In obtaining the formula $3n + 1$, for instance, five prospective teachers first constructed a two-column table showing the number of squares in the first column and number of sticks in the second column. (See **fig. 2** for Raina's solution.) Then they obtained the common difference between two successive values in the second column, wrote $3n$, and observed that each term was "always 1 more than 3 times n." However, when prompted to explain what the coefficient and the constant stood for, they could not explain them within the context of the problem. The numbers 3 and 1 had no other value for them except that they were instrumental in generating all the numbers in the sequence.

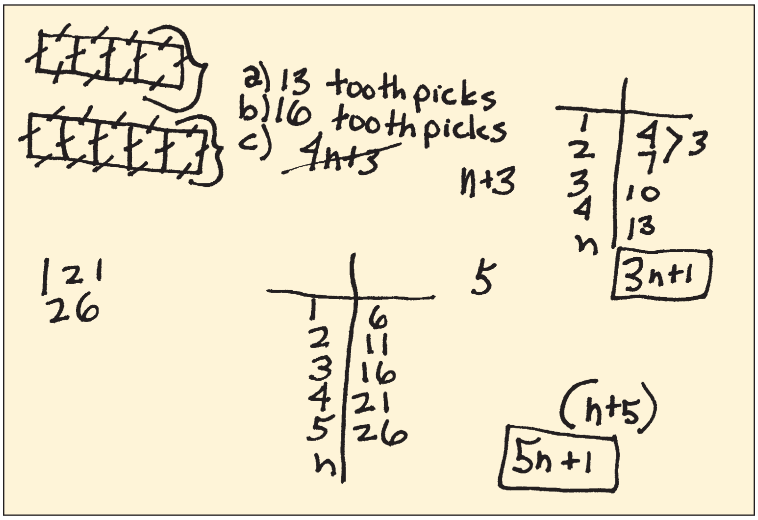


Fig. 2 Raina's numerical solution using finite differences

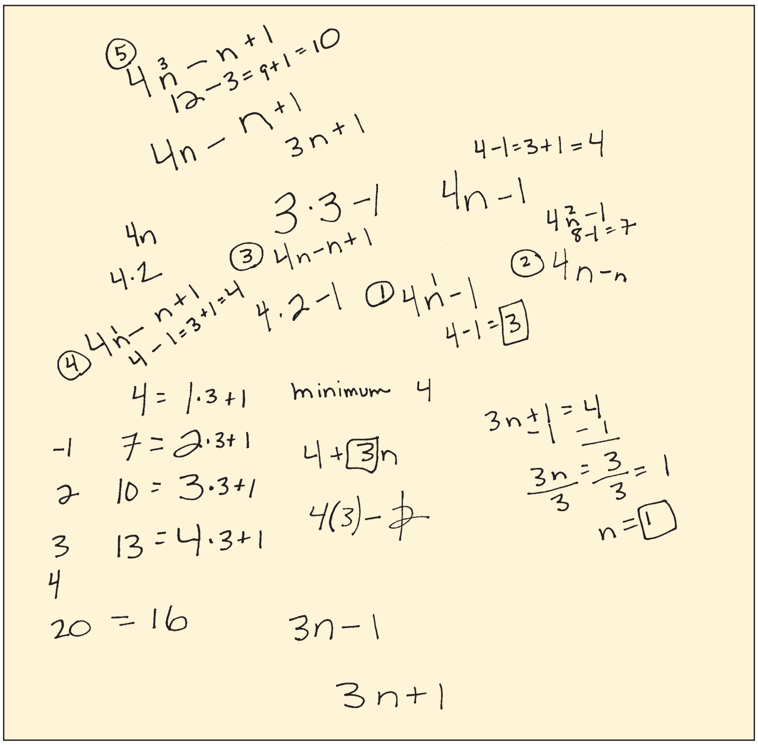


Fig. 3 Jose's numerical solution using trial and error

Trial and error

Jose's method involved *trial and error*. We provide some detail about his thinking process to demonstrate how such a method, although encouraged as a good problem-solving numerical tool, could lead to incorrect conclusions without carefully taking into account all assumptions and if it is justified on the basis of a *mere appearance match* (Gentner 1989). We show his solution in **figure 3** to illustrate the actual steps he used to obtain a numerical generalization. In finding $3n + 1$, he started with the expression $4n - 1$ and computed the value for $n = 1$. Because the value obtained was 3, he then tried $4n - n$ and evaluated this expression for $n = 1$. Seeing that he needed 1 more to obtain the first term, 4, he added 1 to $4n - n$. Once again he evaluated $4n - n + 1$ and saw that it worked for $n = 2$ and $n = 3$. When asked to justify the form $4n - n + 1$, he reasoned as follows:

> OK. Well I see here that four toothpicks equals one square. So it will just keep doubling down to 8 but it doesn't show us here. It's not. 1, 2, 3, 4, 5, 6, 7 'cause you're using that 1. So that's 2 [referring to the two squares]. So what I've tried to do is just go through like a shortcut and cheat 'cause I want 4, 8 but I know I just had to take 1 away to get 7. So now I jumped ahead to 12. OK, but I know it's not because I have to subtract maybe 1 or 2. So 1, 2, 3, 4, 7, 8, 9, 10. So here I subtracted 1 [referring to the first case], here I subtracted 2 [referring to the second case]. Now with three squares, I have to subtract 3 from what normally would make up one square. So it will be 16 subtracted by 3, 13, and I'll try that out. OK and here [referring to the third case] I have 10, 11, 12, 13. So what I didn't know how to do was how to keep saying if I keep adding on squares I have to subtract that many. So it's like taking away 1 from 8, 2 from 12, 3 from 16 . . . so that's why I have this [referring to $4n - n + 1$].

For Jose, the form $4n - n + 1$ actually involved two subgeneralizations, that is, $4n$ and $n + 1$. Each square required four toothpicks, which explains the part $4n$. He then saw the need to subtract 1, 2, 3, . . . , $(n + 1)$ toothpicks in each case beginning with the

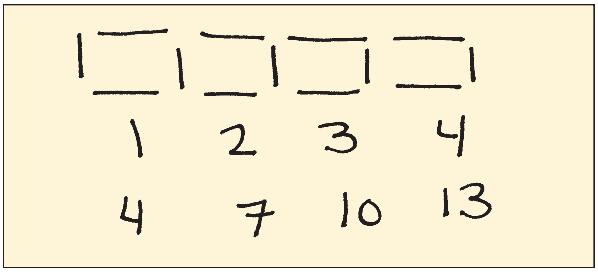


Fig. 4 Shelly's work

second case. Jose, however, was not aware that $4n - (n + 1)$ was not equivalent to $4n - n + 1$. He then simplified the latter expression, obtaining $3n + 1$, then checked to see if the numbers 4, 10, 13, and 16 could be drawn from the rule. Later, when he was asked if it was possible to find a different solution to the same problem, he explained:

> OK it has to be a minimum of 4 toothpicks to be n squares. So . . . so it's a minimum of 4 . . . 4 toothpicks . . . but it could go on forever. But you need something like what we were doing. . . . So hold on. [He writes $4 + 3n$.] I just wrote that to try it out. 4 toothpicks plus 3 times n. OK now I'm starting to think minus 1 since as I keep adding squares on I have to keep subtracting. . . . I can see maybe 4 toothpicks but I'm multiplying. See I don't know what the 3s are there for.

Generalizing Figurally

THE REMAINING SIXTEEN OF FORTY-TWO prospective teachers that we interviewed were predominantly inclined to be more figural than numerical. We also have found that they were more successful at justifying the closed forms they developed. In fact, they could perceive relationships among the available figural cues. The formulas they produced were a clear indication of how they interpreted the figures drawn, including the ones they were asked to construct. The generalizations they developed captured the process of constructing subsequent figures that remained uniform and invariant throughout. Shelly, for instance, first computed the common difference, 3, and then explained that 3 was the number that determined the "difference between one figure to the next," since forming a new square meant adding 3 new sticks. (See **fig. 4** for Shelly's written work.) Without making the generalizing process very complicated for herself, she justified in clear terms what the formula $1 + 3n$ meant in the following manner:

> You're trying to make a full square with 4 toothpicks and if you already have one side then you would be adding 3 more on to it depending on the number of squares that you wanna make 'cause that's how many you would put, that's how many 3s you would add on.

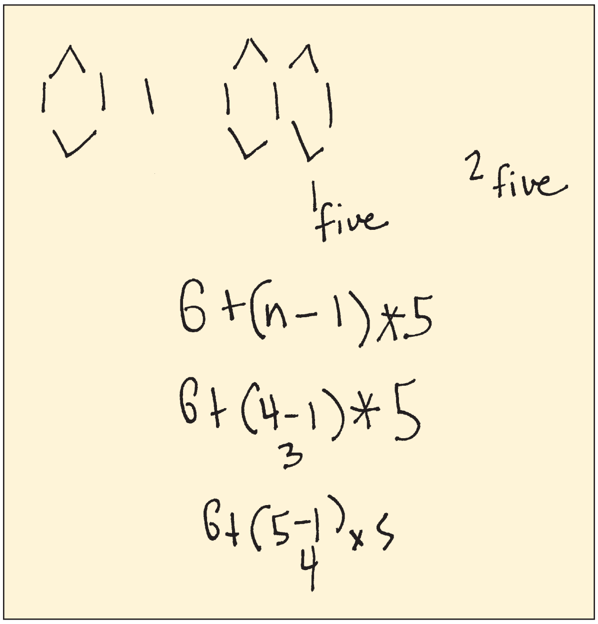


Fig. 5 Chuck's written work on the hexagon problem

We also found it interesting that those who generalized figurally clearly understood the role that symbols played in expressing generalized relationships in explicit terms. In the transcript below, Chuck explains how he initially thought about the formula he developed for the squares task:

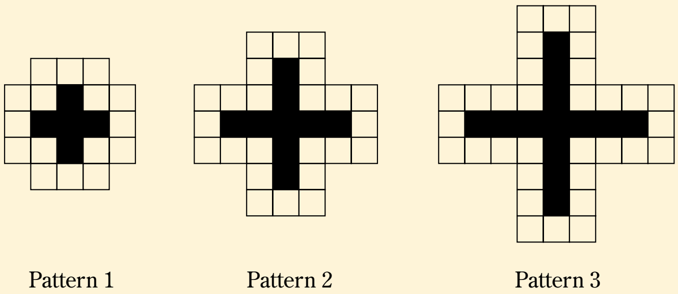
> How many toothpicks are needed to form four squares? So I'm looking for a pattern. For every square, you add 3 more. So let's see. So that would be 4 plus 3 for 2 squares. Plus 3 more would be for 3 squares. So it's 10 toothpicks. So you have 4. So there would be 13. So 13 plus 3 more is 16. So for 3 squares, it would just be two 3s. So there'd be two 3s, three 3s is for 4 squares, and four 3s for 5 squares. For n squares, it would just be n minus 1 [of the] 3s.

Figure 5 illustrates how Chuck applied the way in which he performed inductive reasoning on the squares with the hexagons task.

Implications for Teaching Generalization with Linear Patterns

IN THE PRECEDING SECTIONS, WE DISCUSSED the generalizing strategies of prospective elementary and middle school teachers. Although we did not endeavor to establish a causal link between middle school students' inability to successfully perform generalization and their teachers' generalizing habits, we sought to bring to the surface what prospective teachers in our interview might bring with them when they teach algebra to middle school students. We suspect one reason why many middle school students have difficulty performing generalizations is that many of those who teach them are predominantly more numerical

Tiling Squares Problem



1. How many black tiles are needed to make Pattern 20?

2. How many black tiles are needed to make the nth Pattern?

3. How many white tiles are needed to make Pattern 20?

4. How many white tiles are needed to make the nth Pattern?

Fig. 6 Generating patterns using black-and-white tiles

1. Find a way to count the number of white tiles. Use your method to generalize a pattern for the number of white tiles in the *n*th pattern.

2. Now find another way to count the number of white tiles. Use this new method to generalize a pattern for the number of white tiles in the *n*th pattern.

   Are the two expressions equivalent? Why or why not?

3. Is there a third way to count the number of white tiles? Use this method to generalize a pattern for the number of white tiles in the *n*th pattern. Again, check whether your expressions are all equivalent.

4. From our experience, students have visually counted the number of white tiles for the *n*th pattern in at least four different ways. Yes/No: Can you find more than four? Can you find all of the ways? Show these ways.

Fig. 7 Reflective paper to accompany the generating patterns problem

than figural. Our experiences with students in our classes, both preservice and in-service, confirm this view (Rivera and Becker 2003). Those students who are predominantly numerical usually employ trial-and-error and finite differences as strategies for developing closed forms or partially correct recurrence relations with hardly any sense of what the coefficient and the constant in the linear pattern represent. They see variables as mere placeholders and as generators for linear sequences of numbers. Those who are predominantly figural employ visual strategies in which the focus is on identifying invariant relationships from among the figural cues given. For them, variables move beyond their placeholder function as they are interpreted within the context of a functional relationship. It is interesting to note as well that those who fail to generalize tend to start out with numerical strategies. But because they lack the flexibility to try a figural understanding of the linear patterns, they get stuck and cannot think of alternative ways of generating a generalization beyond what they can manipulate numerically.

The results of our interviews with the forty-two prospective teachers who will eventually teach our children mathematics, including findings we have obtained from our work with in-service teachers, reveal that there is much work that needs to be done. The primary concern is to provide them with a different, albeit meaningful, mathematical knowledge base for teaching generalization effectively at the middle school level other than what they already have in their repertoire of skills. We conclude with two recommendations for better classroom practice.

1. *Teachers need to give their students activities and problem situations that de-emphasize the numerical and emphasize a figural understanding of generalization.* An activity is illustrated in **figure 6**. Good traditional mathematical practice promotes a hierarchic view of algebraic thinking or reasoning that shifts from the perceptual to the conceptual, from the informal to the formal, from the concrete to the abstract, and from figural to numeric. That said, we believe that a dynamic view is more productive in the long run so that students are able to oscillate between two modes or approaches, enabling them to develop greater flexibility, notational fluency, and representational competence. Drawing on our work with elementary and middle school teachers, we note that figural generalizers have a more meaningful understanding of the numerical strategies they construct, and that numerical generalizers are oftentimes unable to *see* patterns and justify their formulas. Further, the ones who are adept at figural generalization could see through invariant properties, relations, or attributes visually, and they could explain the significance of *y*-intercepts and slopes as rates of change in concrete, tangible terms.

2. *Middle school students stand to benefit from a multiple representational view of generalization in both form and approach.* The reflective paper in **figure 7** provides an example of an activity that addresses this recommendation. Students could be asked to obtain several different expressions figurally for the number of white tiles. Further, they could be encouraged to explore what relationships

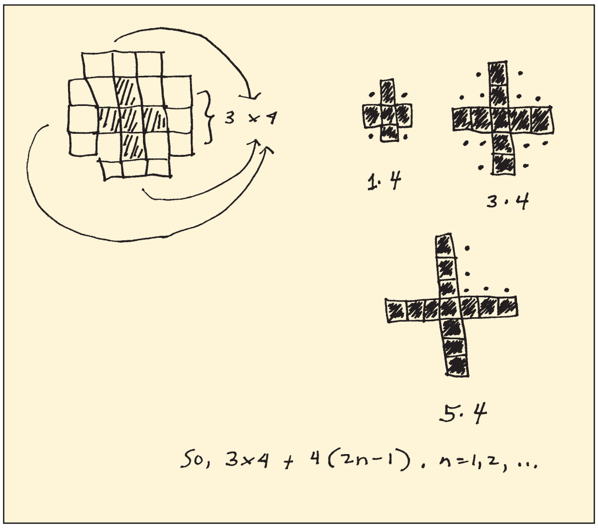


Fig. 8 Figural explanation for $3 \times 4 + 4(2n - 1)$, where n is a natural number

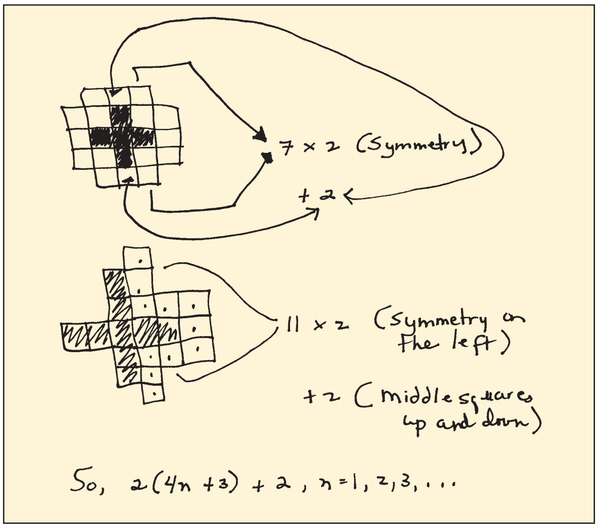


Fig. 9 Figural explanation for $2(4n + 3) + 2$, where n is a natural number

are possible between and among the different expressions. This task naturally introduces students to the notion of *equivalence*, a central concept in algebra and in all areas of mathematics. In the given activity, because students are counting the same number of white tiles for the nth pattern, the different formulas, despite the differences in form, could be taken as equivalent and justified in figural terms. For instance, the following two expressions are equivalent:

1. $3 \times 4 + 4(2n - 1)$, where n represents "pattern number." (See **fig. 8** for a visual explanation.)
2. $2(4n + 3) + 2$, where n represents "pattern number." (See **fig. 9**.)

Teachers also need to help students establish connections between figural and numerical strategies, which could be done by encouraging students to discuss multiple representations of generalizations. Students need to be aware that different paths and viable solution approaches can lead to several different formulas.

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