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ORIGINAL ARTICLE

Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns

F. D. Rivera · Joanne Rossi Becker

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Abstract This paper discusses the content and structure of generalization involving figural patterns of middle school students, focusing on the extent to which they are capable of establishing and justifying complicated generalizations that entail possible overlap of aspects of the figures. Findings from an ongoing 3-year longitudinal study of middle school students are used to extend the knowledge base in this area. Using pre-and post-interviews and videos of intervening teaching experiments, we specify three forms of generalization involving such figural linear patterns: constructive standard; constructive nonstandard; and deconstructive; and we classify these forms of generalization according to complexity based on student work. We document students' cognitive tendency to shift from a figural to a numerical strategy in determining their figural-based patterns, and we observe the not always salutary consequences of such a shift in their representational fluency and inductive justifications.

1 Introduction

Research on patterning and generalization over the past decade or so has empirically demonstrated the remarkable, albeit fundamental, view that individuals tend to see the same pattern P differently. Consequently, this means they are likely to produce different generalizations for P. For example, when we asked 42 undergraduate K-8 pre-service teachers to establish a general formula for the total number of matchsticks at any stage in the *Adjacent Squares Pattern* shown in Fig. 1, Chuck obtained his generalization "$4 + (n - 1)3$" in the following manner:

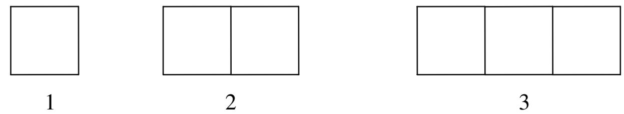
> How many matchsticks are needed to form four squares? So ahm I'm looking for a pattern. For every square you add three more. So let's see. So that would be 4 plus 3 for two squares. Plus 3 more would be for three squares. So it's 10 matchsticks. So you have 4. So there would be 13. So 13 plus 3 more is 16. … So, for three squares, it would have to be two 3s. So there'd be two 3s. Three 3s is for four squares, and four 3s for five squares. For n squares, it would just be ahm n minus one 3s. (Rivera & Becker, 2003, p. 69).

When we gave the same pattern in Fig. 1 to a group of middle school students three times over a 2-year period, first when they were in sixth grade and then twice in seventh grade, all of their generalizations consistently took the form $T = (n \times 3) + 1$. For example, in a clinical interview prior to the Year 2 teaching experiment, Dung, in seventh grade, initially set up a two-column table of values, listed the pairs (1, 4), (2, 7), and (3, 10) and noticed that "the pattern is plus 3 [referring to the dependent terms]." He then concluded by saying, "the formula, it's pattern number $\times$ 3 plus 1 equals matchsticks," with the coefficient referring to the common difference and the y-intercept as an adjustment value that he saw as necessary in order to match the dependent terms. When he was then asked to justify his formula, he provided the following faulty reasoning in which he projected his formula onto the figures in a rather inconsistent manner:

> For 1 [square], you times it by 3, it's 1, 2, 3 [*referring to three sides of the square*] plus 1 [*referring to the*

F. D. Rivera (✉) · J. R. Becker
Department of Mathematics, San José State University,
San Jose, CA 95192-0103, USA
e-mail: rivera@math.sjsu.edu

Square Toothpicks Pattern. Consider the sequence of toothpick squares below.



A. How many toothpicks will pattern 5 have? Draw and explain.

B. How many toothpicks will pattern 15 have? Explain.

C. Find a direct formula for the total number of toothpicks T in any pattern number n. Explain how you obtained your answer.

D. If you obtained your formula numerically, what might it mean if you think about it in terms of the above pattern?

E. If the pattern above is extended over several more cases, a certain pattern uses 76 toothpicks all in all. Which pattern number is this? Explain how you obtained your answer.

F. Diana's direct formula is as follows: $T = 4 \cdot n - (n - 1)$. Is her formula correct? Why or why not? If her formula is correct, how might she be thinking about it? Who has the more correct formula, Diana's formula or the formula you obtained in part C above? Explain.

Fig. 1 The adjacent squares pattern task in compressed form

> *left vertical side of the square*]. For pattern 2, you count the outside sticks and you plus 1 in the middle. For pattern 3, there's one set of 3 [*referring to the last three sticks of the third adjacent square*], two sets of 3 [*referring to the next two adjacent squares*] plus 1 [*referring to the left vertical side of the first square*].

We also found it interesting that none of the middle school students came up with a general form similar to Chuck's. Further, when they were asked to explain an imaginary student's formula, $T = 4n - (n - 1)$, for the Square Toothpicks Pattern (Fig. 1) in Year 1 of the study, prior to a teaching experiment on constructive and deconstructive generalization, they found this and other similar tasks difficult.

In this article, we take the tack of extrapolating issues relevant to the following two questions: what is the nature of the content and structure of generalization involving figural patterns of middle school learners (i.e., Grades 6–8, ages 11–14)? To what extent are they capable of establishing and/or justifying more complicated generalizations? In addressing the first question, we initially survey relevant research in the area of middle school algebraic thinking and then consider how findings in our ongoing longitudinal research at the middle grades in relation to generalization further confirm and/or extend the current knowledge base in the area. The second question zeroes in on what the middle school children in our 3-year study could accomplish within the scope of their competence, including, and especially, factors that inhibit them from constructing and/or justifying more complicated algebraic generalizations. For example, how is it that adults like Chuck could easily generate a general form, or seem to exhibit pattern flexibility, which many, if not, most middle school students like Dung could not easily, or might never, accomplish? Are middle school students simply developmentally underprepared to produce such forms, or can they acquire Chuck's process through more learning (i.e., more experience)? Finally, our overall intent in raising the two issues above is to initiate a complicated conversation on possibly comparable, as well as different, cognitive characteristics between middle school and elementary (or early) algebraic thinking in relation to patterning and generalization involving figural cues. For example, are there similarities and/or differences in the way elementary and middle school children establish invariant properties or relationships among the figural cues in a pattern? Do both groups share similar levels of expressing a generality involving figural cues? Are middle school children more capable of perceptual agility in patterning than elementary school children?

2 Recognizing regularities in patterns

Several researchers have pointed out that the initial stage in generalization involves "focusing on" or "drawing attention to" a possible invariant property or relationship (Lobato, Ellis, & Muñoz, 2003), "grasping" a commonality or regularity (Radford, 2006), and "noticing" or "becoming aware" of one's own actions in relation to the phenomenon undergoing generalization (Mason, Graham, & Johnston-Wilder, 2005). Lee (1996) poignantly surfaces the central role of "perceptual agility" in patterning and generalization which involves "see[ing] several patterns and [a] willing[ness] to abandon those that do not prove useful [i.e., those that do not lead to a formula]" (p. 95). Mason et al. (2005) points out as well how specializing on a particular case in a pattern on the route to a generalization necessitates acts of "paying close attention" to details, especially those aspects that change and/or stay the same, best summarized in Mason's (1996) well-cited felicitous phrase of "seeing the general through the particular." Results of our earlier work with 9th graders (Becker & Rivera, 2005) and undergraduate majors (Rivera & Becker, 2007, 2003) also confirm such a preparatory act whereby perception—as a "way of coming to know" an object or some property or fact about the object (Dretske, 1990)—is necessary and fundamental in generalization. Of course, there are other researchers who emphasize the fundamental, genetic role of invariant acting in the construction of an intentional generalization (Dörfler, 1991; Garcia-Cruz & Martinón, 1997; Iwasaki & Yamaguchi, 1997). In this article, we pursue the cognitive perception perspective in patterning.

Especially in the case of patterning tasks that involve figural cues, we note that among the most important perception types that matter is visual perception. Visual perception involves the act of coming to see; it is further characterized to be of two types, namely, sensory perception and cognitive perception. Sensory (or object) perception is when individuals see an object as being a mere object-in-itself. Cognitive perception goes beyond the sensory when individuals see or recognize a fact or a property in relation to the object. For example, young children who see consecutive groups of figural cues such as the *Adjacent Squares Pattern* in Fig. 1 as mere sets of objects exhibit sensory perception. However, when they recognize that the cues taken together actually form a pattern sequence of objects, they manifest cognitive perception. Cognitive perception necessitates the use of conceptual and other cognitive-related processes, enabling learners to articulate what they choose to recognize as being a fact or a property of a target object. It is mediated in some way through other types of visual knowledge that bear on the object, and such types could be either cognitive or sensory in nature. In the rest of the article, we address issues relevant to middle school students' cognitive perceptions of figural-based patterns. Foregrounding cognitive perception in pattern formation and in the interpretation of a generalization, in fact, has allowed us to investigate how the students see aspects of patterns they find relevant which consequently influence the content and structure of generalizations they produce, including elements that constitute the structure of their cognitive perception in relation to these special types of objects.

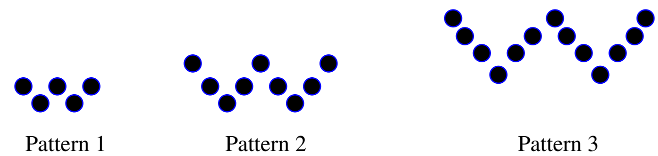
When Duval (1998) claims that "there are various ways of seeing a figure" (p. 39), he is, in fact, referring to a cognitive perception of the figure. Duval identifies at least two ways in which learners manifest their recognition of the figure, that is, perceptual and discursive. Perceptual apprehension involves seeing the figure as a single gestalt. For example, a student might see a quadrilateral in the representational context of a roof or the top part of a table. Discursive apprehension involves seeing the figure as a configuration of several constituent gestalts or as sub-configurations. For example, another student might see the same quadrilateral as consisting of sides that are represented by line segments. The shift from the perceptual—seeing objects as a whole—to the discursive—seeing objects by parts—is indicative of a dimensional change in the cognitive perception of the figure. In relation to figural-based patterns, students who, on the one hand, perceptually apprehend, say, the cues in Fig. 1 might see squares that grow by the stage (for e.g.: stage 1 has one square, stage 2 has two squares, etc.). On the other hand, those who discursively apprehend the same cues might see squares that are produced *either* by repeatedly adding three sides to form a new square, a *constructive generalization*, or by first constructing the appropriate number of squares, multiplying that number by 4 since there are four sides to a square, and finally seeing overlaps (for e.g.: stage 2 has two groups of four sides with an overlapping "interior" side, pattern 3 has three groups of four sides with two overlapping "interior" sides. etc.), a *deconstructive generalization*. Also, Duval (2006) foregrounds the cognitively complex requirements of semiotic representations in both perceptual and discursive domains. Especially in the case of patterning in algebra, because there are many different ways of expressing a generalization for the same pattern, the primary resolve involves assisting learners to recognize the viability and equivalence of several generalizations that are drawn from several "semiotic representations that are produced within different representation systems" (p. 108). For example, Dung (see Figs. 2, 3) obtained his general formula by initially manipulating the corresponding numerical cues that he later justified figurally, while Chuck (see Fig. 4) established his formula from the available figural cues. Both learners operated under two different representational systems and, thus, produced two different, but equivalent, direct expressions for the same pattern.

3 Methodology

In Fall 2005 and Fall 2006, the first author collaborated with two middle school mathematics teachers in

Fig. 2 W-dot pattern task in compressed form

W-Dot Sequence Problem. Consider the following sequence of W-patterns below.



A. How many dots are there in pattern 6? Explain.

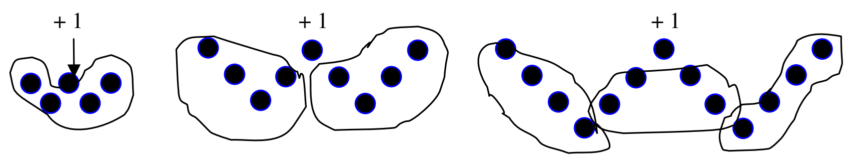
B. How many dots are there in pattern 37? Explain.

C. Find a direct formula for the total number of dots D in pattern n. Explain how you obtained your answer. If you obtained your formula numerically, explain it in terms of the pattern of dots above.

D. Zaccheus's direct formula is as follows: $D = 4(n + 1) - 3$. Is his formula correct? Why or why not? If his formula is correct, how might he be thinking about it? Which formula is correct: your formula or his formula? Explain.

E. A certain W-pattern has 73 dots altogether. Which pattern number is it? Explain.

Fig. 3 Dung's figural justification of the W-dot pattern in Fig. 2



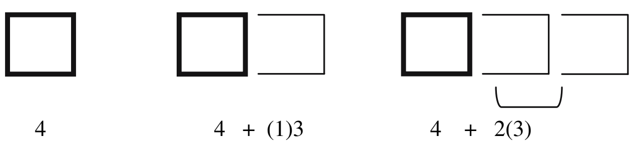


Fig. 4 Chuck's constructive generalization for the pattern in Fig. 1

developing and implementing two related design-driven teaching experiments that involve patterning and generalization in algebra. Pre- and post-clinical interviews with the participating students were also conducted by the second author. Learnings from the pre-interviews were incorporated in the evolving teaching experiments with the participants, and the post-interviews were meant to assess students' abilities to establish and justify their generalizations, including the extent of influence of classroom practices in their developing capacity to generalize. In Fall 2005, the sixth-grade class consisted of twenty-nine students (12 males, 17 females, mean age of 11). In Fall 2006, three students moved to a different school and were replaced with six new students.

Intrinsic to classroom teaching experiments that employ design research are two objectives, that is, developing an instructional framework that allows specific types of learning to materialize and analyzing the nature and content of such learning types within the articulated framework. Thus, in every design study, theory and practice are viewed as being equally important, which includes rigorously developing and empirically justifying a domain-specific instructional theory relevant to a concept being investigated. Further, the content of the proposed instructional theory involves a well-investigated learning trajectory and appropriate instructional tools that enable student learning to take place in various phases of the trajectory. Finally, instruction in design studies is characterized as having the following elements: it is experientially real for students; it enables students to reinvent mathematics through, at least initially, their commonsense experiences, and; it fosters the emergence, development, and progressive evolution of student-generated models.

One instructional objective of the classroom teaching experiments on patterning and generalization involves providing students with an opportunity to engage in problem-solving situations that would enable them to meaningfully acquire the formal mathematical requirements of algebraic generalization. We both were cognizant of the fact that our students' thinking could possibly go through generalizing stages or levels perhaps similar to the ones that the students in Radford's (2006) study went through. We were also aware that some generalizing situations that our students would tackle would be real and others *experientially real* in the sense that they would not need to actually experience such situations but "should be able to imagine acting" in them (Gravemeijer, Bowers, & Stephan, 2003, p. 52). Considering the fact that we both shared Freudenthal's (1991) views about the nature of mathematics and mathematical activity, it was not difficult for us to subscribe to the instructional theory of Realistic Mathematics Education (RME). In RME, learners use *models of* their informal mathematical processes to assist them to develop *models for* more formal processes, and being able to successfully transition is an indication that they are constructing a new mathematical reality. Formalizing is, thus, seen as "growing out of their mathematical activity" and mathematizing, more generally, involves "expanding [their] common sense" with the same reality as "experiencing" in everyday life (Gravemeijer & Doorman, 1999, p. 127).

In the Fall 2005 experiment, we used two algebra units in the Mathematics-in-Context (MiC) curriculum that adhere to RME. Also, taking note of the algebra requirements of the California state standards for sixth graders, we selected sections from the units *Expressions and Formulas* (Cole & Burrill, 2006) and *Building Formulas* (Burrill, Cole, & Pligge, 2006) that became the basis of a three-phase classroom teaching experiment on algebraic generalization. The two algebra units, which correspond to the first two phases in the teaching experiment, contained activities that fostered the development of algebraic generalization through a series of horizontal and vertical mathematization tasks. According to Treffers (1987), horizontal mathematization involves transforming real and experientially real problems into mathematical ones by using strategies such as schematizing, discovering relations and patterns, and symbolizing. Vertical mathematization involves reorganizing mathematical ideas using different analytic tools such as generalizing or refining of an existing model. In both units, students first explored horizontal activities that allowed them to build an informal mathematical model. They then engaged in vertical activities.

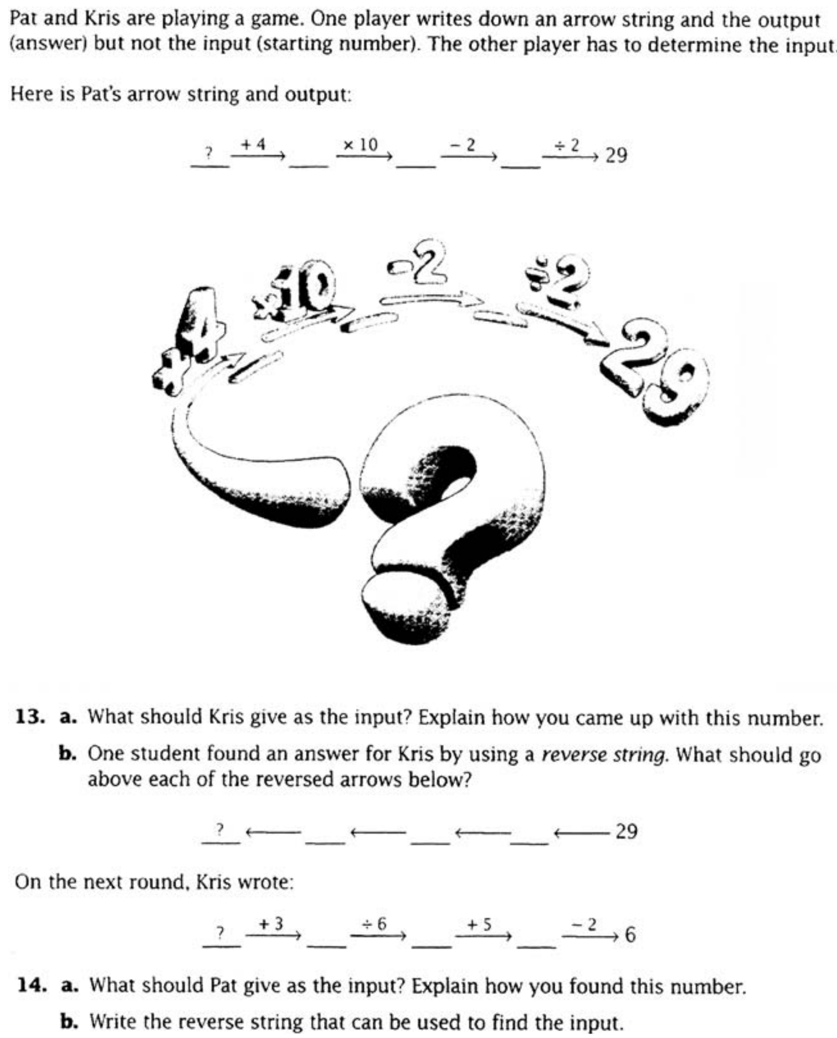
In the *Expressions and Formulas* unit (Cole & Burrill, 2006), each section had the students starting out with a problem situation that involved using an arrow language notation to initially organize the situation and later to express relationships between two relevant quantities. An example is shown in Fig. 5. The arrow notation was meant to articulate the different numerical actions and operations that were needed to carry out a string of calculations in an activity. Also, the task situations were either stated in words or accompanied by tables, and they contained items that necessitated either a straightforward or a reverse calculation. The *Patterns* section in the *Building Formulas* unit (Cole, Burrill & Pligge, 2006) was the only one that we used in the teaching experiment because we were working within the stipulated sixth-grade algebra requirements of the state's official mathematics framework. In this section, arrow language was employed less in favor of recursive formulas and direct formulas in closed, functional form. The students were asked to deal with problem situations that consistently contained the following tasks relevant to generalizing: extending a near generalization problem physically (for example: drawing or demonstrating with the use of available manipulatives) and/or mentally (reasoning about it logically); calculating a far generalization[1] task using either a figural or a numerical strategy; developing a general formula recursively and/or in closed, functional form, and; solving problems that involve inverse or reverse operations. In all problem situations, tables were presented and employed as an alternative representation for organizing the given data. Finally, students dealt with tasks that asked them to reason and to make judgments about the equivalence of several different formulas for the same problem situation.

In the third phase of the teaching experiment, we asked the students to work through several de-contextualized patterning problems whose basic structure was similar to the ones that have been described in the paragraph above (see, for e.g., Fig. 2). Also, we developed problems that necessitated the students to develop both numerical and figural generalization. We note that in all generalizing situations that the students had to deal with, we required them to establish and justify their constructed generalizations. We saw the justification of their generalizations to be equally as important as their generalized statements. Of course, justification could mean many things (cf. Lannin, 2005), however, considering the cognitive level of the students who have just begun learning domain-specific knowledge and practices in algebra, we more or less confined the notion of justification to their capacity to reason, in the sense following Hershkowitz (1998), "to understand, to explain, and to convince" (p. 29). In fact, we share Lannin's (2005) perspective, which he demonstrated in his work with 25 US sixth graders, when he pointed out how justification seems to have been relegated to the "realm of geometric proofs" when, in fact, students' justifications in the context of generalization could "provide a window for viewing the degree to which they see the broad nature of their generalizations and their view of what they deem as a socially accepted justification" (p. 232).

In the Fall 2006 three-phase experiment, the seventh-grade class used *Building Formulas* and portions of *Patterns and Figures* (Spence, Simon, & Pligge, 2006) in the first two phases, with the third phase the same as in the description above.

[1] Consider a sequence function f: $n \rightarrow$ R whose domain is the set of natural numbers. We arbitrarily set our far generalization task to be those cases where $n \geq 10$.

Fig. 5 Arrow string example (Cole & Burrill, 2006, p. 28)


Pat and Kris are playing a game. One player writes down an arrow string and the output (answer) but not the input (starting number). The other player has to determine the input.

Here is Pat's arrow string and output:

$\underline{\ ?\ } \xrightarrow{+4} \underline{\quad} \xrightarrow{\times 10} \underline{\quad} \xrightarrow{-2} \underline{\quad} \xrightarrow{\div 2} 29$

13. a. What should Kris give as the input? Explain how you came up with this number.

b. One student found an answer for Kris by using a *reverse string*. What should go above each of the reversed arrows below?

$\underline{\ ?\ } \longleftarrow \underline{\quad} \longleftarrow \underline{\quad} \longleftarrow \underline{\quad} \longleftarrow 29$

On the next round, Kris wrote:

$\underline{\ ?\ } \xrightarrow{+3} \underline{\quad} \xrightarrow{\div 6} \underline{\quad} \xrightarrow{+5} \underline{\quad} \xrightarrow{-2} 6$

14. a. What should Pat give as the input? Explain how you found this number.

b. Write the reverse string that can be used to find the input.

4 Constructive versus deconstructive generalizations

A *constructive generalization* of a linear figural pattern refers to those direct or closed degree-1 polynomial forms which learners easily abduce or induce from the available cues as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent gestalts or parts (see Rivera & Becker, 2007, for a discussion involving abduction and induction). We classify Dung's formula, $D = n \times 4 + 1$, for the *W-dot Pattern* in Fig. 2 as a direct constructive generalization that exhibits the standard linear form $y = mx + b$ (see Fig. 3). Chuck's formula for the pattern in Fig. 1, on the other hand, is a direct one with an equivalent nonstandard linear form (Fig. 4). A *deconstructive generalization* of a linear figural pattern applies in cases in which learners generalize on the basis of initially seeing overlapping sub-configurations in the structure of the cues. Consequently, the corresponding general linear form involves a combined addition-subtraction process of separately counting each sub-configuration and taking away parts (sides or vertices) that overlap. For example, the general form $T = 4n - (n - 1)$ for the pattern in Fig. 1 involves counting squares, multiplying them by 4, and then taking away the overlapping interior sides (Fig. 6). Thus, while both constructive and deconstructive generalizations yield a direct, closed generalized formula, only those we call deconstructive involve visualization with overlapping of sections of the pattern, yielding a formula including subtraction of those portions counted twice (Table 1).

Several research studies at the middle school level have provided sufficient evidence that shows learners' proclivities towards producing more constructive than

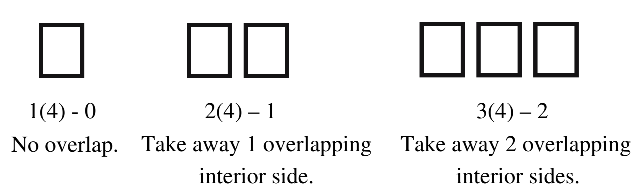


Fig. 6 A deconstructive generalization for the pattern in Fig. 1

Table 1 Summary of problems and sample constructive and deconstructive generalizations

Figure #	Possible constructive generalization	Possible deconstructive generalization
1: Square toothpick	$3(n - 1) + 4$; $2n + (n + 1)$	$4n - (n - 1)$
2: W dot	$4n + 1$	$4(n + 1) - 3$
7: Circles	$(n + 1) + (n + 3)$	$2(n + 3) - 2$
8: Losing squares	$-2n + 34$	$32 - 2(n - 1)$
9: Triangular toothpick	$2n + 1$	$3n - (n - 1)$
10: Circle dots	$n + (n - 1)$	$2n - 1$
14: Two layer circles	$2n + 3$	$(2n + 2)2 - (2n + 1)$

deconstructive generalizations. For example, when Robertson and Taplin (1995) asked 40 Australian 7th graders to establish a generalization for the pattern sequence in Fig. 1, while none of them could state an algebraic generalization, their incipient generalizations took the form of direct and standard constructive verbal statements. Seven students perceived four toothpicks that pertained to the original square in stage 1 and the repeated addition of three toothpicks each time from stage to stage. There were eight students who offered the nonstandard verbal constructive generalization, $3(n - 1) + 4$, although none offered an articulation that was as clear as Chuck's. Only one student began to think about the pattern in a deconstructive way; however, the student was not able to figure out how many toothpicks to take away despite seeing the pattern as consisting of overlapping squares. When the same problem was given to a cohort of four hundred thirty 12- to 15-year-old Australian students, findings from English and Warren's (1998) study also showed that, among the less than 40% of students who successfully obtained a generalization, they expressed their generalities on this and other patterning tasks in constructive terms similar to what Robertson and Taplin (1995) found. For example, a student developed the general expression $2x + (x + 1)$, where $2x$ refers to the top and bottom row sticks and $(x + 1)$ to the column sticks, after seeing two invariant properties within and across cues.

4.1 Origins of factual generalization

So, while descriptions of constructive generalities for figural patterns abound, the more important question involves the epigenesis of such types of generalization—that is, how does constructive objectification come about? First, Radford (2003) notes that there are different semiotic means of objectification in relation to pattern cues, that is, possibly different ways in visibly surfacing attributes and properties of, or relationships among, cues with the use of signs and relevant processes or operations. Second, Radford (2003, 2006) advances the view that there are at least three layers of algebraic generalization—factual, contextual, and symbolic—based on his 3-year longitudinal work with middle and junior high school students. Third, purposeful instruction through well-designed classroom teaching experiments could scaffold the development of closed forms of constructive generalizations in middle school children (Lannin, Barker, & Townsend, 2006; Martino & Maher, 1999; Steele & Johanning, 2004). In the following paragraphs, we dwell on cognitive-related issues at the entry stage of generality, that is, factual, since both contextual and symbolic layers are marked indications of further essentializing and increasing formality on the basis of the stated factual expressions.

At the pre-symbolic stage of factual generalizing involving increasing linear patterns, students often start with a recursive relation that is both additive and arithmetical in nature. As a matter of fact, studies done in different settings (for e.g., countries) and in different contexts (prior to formal instruction in algebra, during and/or after a teaching experiment, etc.) with middle school children have asserted the use of recursion as the entry (and, in some cases, the final) stage in factual generalizing (Becker & Rivera, 2006a, b; Bishop, 2000; Lannin, Barker, & Townsend, 2006; Orton, Orton, & Roper, 1999; Radford, 2003; Sasman, Olivier, & Linchevski, 1999; Swafford & Langrall, 2000). For example, in the case of increasing figural sequences, it is usually easy to first perceive the dependent terms as constantly being increased by a common difference. As soon as this takes place, students' thinking is then characterized in two ways. First, they see two consecutive cues as being different and, using the method of "differencing" (Orton & Orton, 1999, p. 107), the same number of objects is constantly being added from one cue to the next, leading to a recursive, arithmetical generalization (of the type $u_n = u_{n-1} + c$, where c is the common difference). Then, some students further develop emergent factual generalizations from the arithmetical generalization. Two possible factual generalizations involving the pattern in Fig. 1 are as follows: $4 + 3 + 3 + 3 + \ldots$; $1 + 3 + 3 + 3 + \ldots$ Second, a structural similarity is observed among and, thus, connects two or more cues in a relational way. Especially in the case of increasing linear patterns that visually demonstrate growth, constructing a succeeding cue from a preceding one often involves a straightforward process of simply adding a constant number of objects on particular locations of the preceding cue. That is, the basic structure of the unit figure is perceived to stay the same despite the fact that equal amounts of objects are conjoined in various parts of the figure in a particular, predictable manner. Such method of construction does not necessitate making a figural change (in Duval's 1998 sense) on the part of the learner. Also, it seems to be the case that almost all linear patterns that exhibit growth tend to be "transparent" in the sense that the closed formulas associated with them are somehow visibly embodied in each cue (following Sasman, Olivier, & Linchevski, 1999). Radford (2003) further notes how in the factual stage of generalizing, invariant acting from one cue to the next operates at the concrete level that eventually leads to the abstraction of a numerical or operational scheme for the figural pattern. Hence, generalizations that have been mediated by such actions tend to be consequentially constructive and almost always standard (whether rhetorical, syncopated, or symbolic in form; cf. Sfard, 1995).

Even with patterning tasks that require middle school children to first specialize on the route to establishing a generality as a consequence of not being provided with the usual consecutive sequence of figural cues (i.e., the initial cases such as the pattern in Fig. 1), middle school children would be predisposed to establishing constructive generalizations. For example, Swafford and Langrall (2000) asked ten middle- to high-math achieving 6th grade students to solve the *borders pattern task* prior to a formal course in algebra. The task began with a drawn 10×10 square grid in which the four borders of the grid are shaded. The students were asked to figure out the total number of squares on the border, and the task was repeated in a 5×5 grid. The students were then asked to describe how to determine the total number of squares in the border of an $N \times N$ grid. Results on this task show that, while none of the students offered a recursive rule, the general verbal descriptions ranged in form from the constructive to the deconstructive. When translated in symbolic form, two of the verbal constructive generalizations obeyed the following forms: (1) $n + n + (n - 2) + (n - 2)$; (2) $n + (n - 1) + (n - 1) + (n - 2)$. We found it interesting that only one student offered a verbal deconstructive generalization that followed the form $4n - 4$. When the above task and other similar ones were given to eight 7th grade students in Steele and Johanning's (2004) study in the context of a problem-solving enriched teaching experiment, only three students came up with deconstructive symbolic generalizations.

4.2 Operations used in developing generalizations

Another relevant epigenetic issue that we also considered in relation to patterning involves the operations that are employed in developing a constructive generalization. For example, in the case of increasing linear patterns, students need to have solid grounding in addition and multiplication of whole numbers. With decreasing linear patterns, they need to know how to manipulate addition, subtraction, and multiplication of integers (cf. English & Warren, 1998; Stacey & MacGregor, 2001). Let us deal with increasing linear patterns first. When 8th graders in Radford's (2002) study were asked to establish a generalization for the *Circles Pattern* in Fig. 7, a group of three students did not immediately suggest a recursive rule, which actually was offered next, because what they perceived first was an additive relationship between the top and the bottom rows ["add 1 at the bottom" and "add 3 on top" which, when expressed in the general case, takes the form $(n + 1) + (n + 3)$]. This situation engenders the question of how is it that middle school children do not seem to easily perceive a deconstructive generalization such as $2(n + 3) - 2$ (i.e., seeing a rectangular array of two rows of circles and then taking away two corner circles in the bottom row)? Do the operations of multiplication and subtraction, which are often employed in stating a deconstructive generalization, play a role?

Gelman and colleagues (Gelman, 1993; Gelman & Williams, 1998; Hartnett & Gelman, 1998) have advanced and empirically justified a rational constructivist account of cognitive development among young children which presupposes the existence of innate or core skeletal mental structures (such as arithmetical structures) that enable them to easily develop and process new information as long as it is consistent with their core structures. Hartnett and Gelman (1998) write:

> As long as inputs are consistent with what is known, then novices' active participation in their learning can facilitate knowledge acquisition. But when available mental structures are not consistent with the inputs meant to foster new learning, such self-initiated interpretative tendencies can get in the way (p. 342).

In the case of middle school children who develop constructive generalizations, perhaps it is the case that their constructive generalizations, which involve using the operations of addition and multiplication of whole numbers, map easily onto their current understanding of

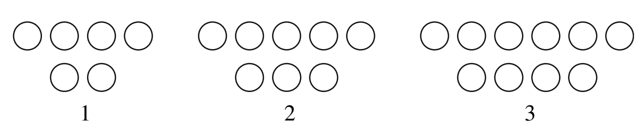


Fig. 7 Circles pattern

what numbers are and how such entities are used, represented, and manipulated. Thus, constructive generalizing will proceed naturally and smoothly. Moreover, middle school children are likely to associate increasing growth patterns with counting objects over several non-overlapping constituent gestalts and then using the addition and multiplication of counting numbers as useful operators in obtaining a final count. Hence, their core arithmetical structures assist in this developing capacity towards making constructive generalizations. This being the case, very few students will apprehend increasing patterns as being embedded in a figural process that involves the operation of subtraction, that is, by utilizing a figural change process of seeing sub-configurations and removing parts that overlap.

Many decreasing linear patterns can also be expressed as constructive generalizations in the form $y = mx + b$, where $m < 0$. The rational constructivist perspective of Gelman and colleagues could be used to explain why many middle school children find generalizations involving negative differencing such a difficult task to accomplish. In Year 2 of our longitudinal study, we saw that the seventh graders' primary cognitive difficulty in seventh grade with decreasing patterns prior to a teaching experiment was how to handle negative differencing and, especially, how to perform operations involving negative and positive integers in which the rules were not consistent with their existing core arithmetical domain (Becker & Rivera, 2007). While we found that they were attempting to "transfer" the existing generalization process they established in the case of increasing linear patterns, they could not, however, make sense of the negative integers and the relevant operations that were used with such types of numbers. For example, Tamara was first asked to establish and justify generalizations for two increasing linear patterns that she accomplished successfully. Her generalizations were constructive and standard, and she was also able to justify the equivalence of several linear forms with the ones she developed. When Tamara was then asked to obtain a generalization for the *Losing Squares Pattern* in Fig. 8, she immediately saw that every stage after the first involves "minusing 2" squares. She then used multiplication to count the total number of squares at each stage. When she then proceeded to obtain a formula, she was perturbed by the negative value of the common difference and said, "I was trying to think of, just like the last time, I was trying to get a formula. … I was thinking of trying to do with the stage number but I don't get it." The presence of the negative difference, including the necessity of multiplying two differently signed numbers, partially and significantly hindered her from applying what she knew about constructing general formulas in the case of increasing patterns. In fact, she had to first broaden her knowledge of multiplication to

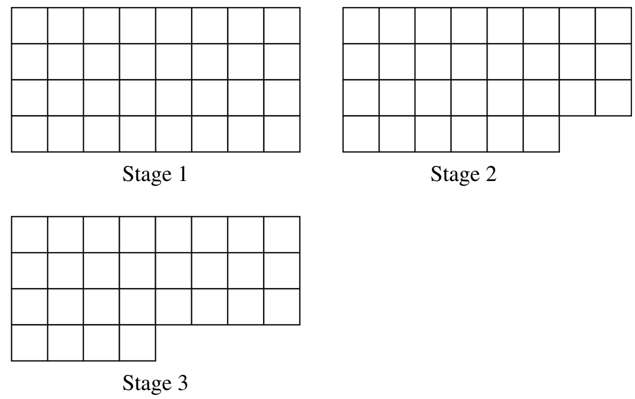


Fig. 8 Losing squares pattern

include two factors having opposite signs before she was finally able to state the form $S = -2 \times n + 34$. Further, while she could explain what the numbers m and b meant in the case of increasing patterns which for her took the constructive form $y = mx + b$, she was unable to justify the forms she established for decreasing linear patterns.

4.3 Factors affecting students' ability to develop constructive generalizations

Even if middle school children are capable of producing more constructive than deconstructive generalizations, there are still other factors that influence their ability to establish a constructive generalization. Language is an important factor (MacGregor & Stacey, 1992; Radford, 2000, 2001; Stacey & MacGregor, 2001). Based on results drawn from Year 7 to Year 10 (ages 12–15) Australian students and their reflections on a national recommendation for a pattern-based approach to algebra, MacGregor and Stacey (MacGregor & Stacey, 1992; Stacey & MacGregor, 2001) surface students' difficulties in "transition[ing] from a verbal expression to an algebra rule" since "students with poor English skills" are often unable to "construct a coherent verbal description" and many of their "verbal description[s] cannot be [conveniently and logically] translated directly to algebra" (MacGregor & Stacey, 1992, pp. 369–370). Stacey and MacGregor (2001) foreground the importance and necessity of the "verbal description phase" in the "process of recognizing a function and expressing it algebraically" (p. 150). Radford (2006) and Küchemann (1981) have also surfaced the problematic status of variable use in students' expressions of generality. In Radford's (2006) layers of algebraic generalization, the presence and use of variables in their proper form and meaning signal the accomplishment of the final stage of symbolic generalization. Radford (2006) notes that while some students may display knowledge of using algebraic language to express a constructive generalization, the variables used in such contexts have to reach their objective state of being subjectified and disembodied placeholders. Küchemann (1981) found that 13 to 15 year-old students tend to interpret variables in patterning situations narrowly in terms of concrete objects rather than as unknown quantities. Radford's (2001) characterization of algebraic language at the layer of symbolic generalizing is best exemplified in the thinking of two small groups of 8th graders on the *Triangular Toothpick Pattern* in Fig. 9 who obtained the generalities $(n + n) + 1$ and $(n + 1) + n$ and perceived them as being different on the basis of having been derived from two different actions. Radford (2001) astutely points out that the use of variables to convey a generality has to evolve. In particular, when students employ a variable in relation to the independent term of the general expression, they need to eventually see that the variable has to shift meaning from being a "dynamic general descriptor of the figures in [a] pattern" to being "a generic number in a mathematical formula" (Radford 2000, p. 255). Thus, their general algebraic language in expressions of generality involves semantically transposing the independent variable from its ordinal character (indexical, positional, deictically based) to the cardinal (as a "number capable of being arithmetically operated" (ibid)).

Another factor that influences middle school children's ability to establish constructive generalizations involves their capacity to use analogies. Since all linear patterns take the constructive general form $y = mx + b$ or some other linear variant, perceiving and using analogies can quickly facilitate the generalizing process. While middle school children are likely to offer a constructive recursive expression, some have been documented to be capable of developing constructive analogical expressions in varying formats even prior to a formal study of algebra and algebraic notation (Becker & Rivera, 2006a, b; Bishop, 2000; Lannin, 2005; Stacey, 1989; Swafford & Langrall, 2000). Performing analogy involves "perceiv[ing] and operat[ing] on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar" (Richland, Holyoak, & Stigler, 2004, pp. 37–38). In Year 1 of our longitudinal study, we identified a possible source of difficulty among the sixth grade students in relation to constructing algebraically useful analogies for particular figural-based patterns. We distinguished between students

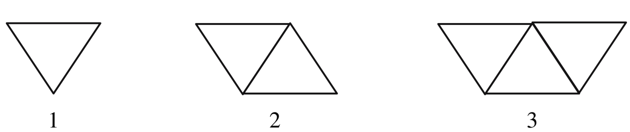


Fig. 9 The triangular toothpick pattern

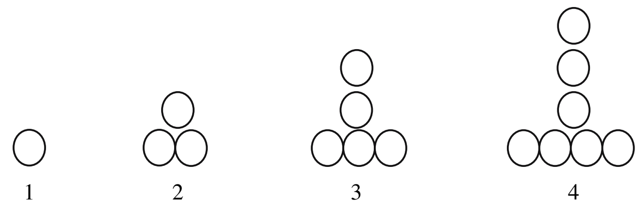


Fig. 10 The circle dots pattern

who perceived and generalized additively from those who employed a multiplicative approach. Those students who used a figural additive strategy, on the one hand, were not thinking in analogical terms at all, and they would frequently employ unit counting from cue to cue. Further, when some of them were provided with manipulatives to copy figural cues that had been drawn on paper, their manipulative-constructed cues did not preserve the structure of individual cues; in fact, they used the manipulatives only as counters. For example, when Dina was asked to obtain a generalization for the total number of dots in the *Circle Dots Pattern* in Fig. 10, her circle chip-based cues in Fig. 11 revealed the extent of her perception of the cues, that is, the cues just kept going up by twos and Dina constructed no particular pattern with the dots as can be seen in Fig. 11. Those who used a figural multiplicative strategy, on the other hand, initially employed analogical reasoning. Employing multiple instead of unit counting, their general statements reflect the invariant structure they thought was evident from cue to cue.

5 A sociogenetic account of the development of two constructive generalization types in a middle school classroom context

In this section, we describe how the middle school students in our ongoing longitudinal work with them established five classroom mathematical practices involving constructive generalization over the course of two teaching experiments on patterning and generalization. The first teaching experiment was conducted in Fall 2005 when they were in sixth grade beginning a formal course in algebra; the second experiment took place in Fall 2006. We note

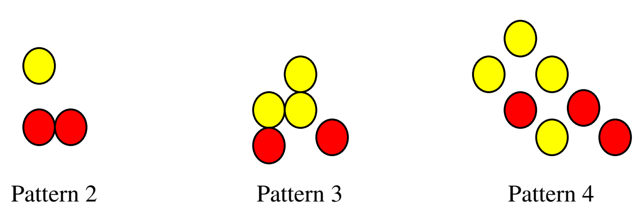


Fig. 11 Dina's interpretation of the circles dots pattern using colored chips

that very few studies at the middle school level have focused on how children develop a generalization practice in socio-genetic terms. Thus, in the narrative that follows, we aim to highlight how certain legitimate mathematical practices could be viewed not as conceptual, received objects that learners simply acquire rather unproblematically but as part of their socio-cultural-developmental transformation drawn and embodied in their activity with other learners.

5.1 Classroom practices in Year 1

In Fall 2005, Mrs. Carrie, a sixth-grade teacher, and the first author implemented a teaching experiment based on two algebra units of the Mathematics-in-Context (MiC) curriculum which provided Mrs. Carrie's class of 29 sixth-grade students (12 males, 17 females, mean age of 11) an opportunity to learn and establish domain-specific classroom mathematical practices relevant to patterning and generalization. Four such practices were constructed and became taken-as-shared in collaborative activity, that is, these practices became the norm for the class as a whole. Two of the practices had their origins in the first MiC unit they used in class (i.e., *Expressions and Formulas*; Cole & Burrill, 2006). First, the students initially employed arrow strings as a method for organizing a sequence of arithmetical operations. They also explored the notion of equivalence through arrow strings that could either be shortened or lengthened depending on the nature of the numbers being manipulated. Second, the use of the arrow strings evolved as the students were asked to deal with more complicated problem situations that were still arithmetical in context. In several more sessions, they developed a connection between constructing an arrow string and a formula in such a way that they used arrow strings as a means of describing invariant operational schemes in the context of generalizing situations. In transitioning from the arrow strings to formulas, the students developed an understanding that a formula, like the arrow strings, consists of a starting number or input, a rule in the form of a sequence of operations, and an output value or expression (see Fig. 5).

Two additional practices emerged when the students began to generalize figural-based patterns that have been initially drawn from the *Patterns* section in the MiC unit *Building Formulas* (Burrill, Cole, & Pligge, 2006). The *third* classroom practice that became taken-as-shared involves generalizing figurally and is exemplified in the classroom episode below in which the students were engaged in developing a formula for the total number of grey and white tiles for new path number n whose figural cues are shown in Figs. 12 and 13. Initially, the students explored specific instances when $n = 3–5$, 9, 15, 30, and 100. In particular, they were not merely asked to obtain the output values but also to describe the patterns without actually drawing them explicitly. The class then generated a recursive rule for each tile type. In the episode below, the discussion that took place between the first author and the class shifted from the recursive rules to the construction of a direct, closed general expression.

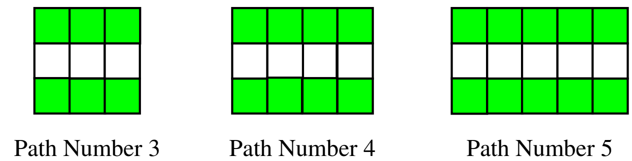


Fig. 12 Urvashi's tile patterns (Burrill, Cole, & Pligge, 2006, p. 2)

FDR: Suppose I want you to describe new path 1,025. That's a big number. I want you to figure out the total number of white and grey tiles for new path 1,025. Emily, how do we do this?
Emily: The whites will be 2,054?
Ford: That's the grey.
Emily: It is?
Ford: Yeah, the white's the middle.
Emily: 1,029.
FDR: Why 1,029?
Emily: Because it's in the middle and in the corners it has four.
FDR: Alright. What about the grey ones? Mark.
Mark: The grey ones are 2,052.
FDR: Why 2,052?
Mark: Because you added the top and the bottom and then you add the two middle.
FDR: Okay, this will be a challenge for some of you. Can you find a formula for me? Suppose, I say, I'm going to use a variable, new path number n. n could mean 1, 2, 3, 4, all the way to 1,025. All the way to a billion.
Dung: n plus 4 equals white.
FDR: Why $n + 4$ equals white?
Dung: Coz n is the number of whites in the middle plus 4 whites on the sides.
FDR: Does that make sense? [*Students* nodded in agreement.] What about the grey ones? The grey ones are a bit more difficult. What's a formula for the number of grey ones?

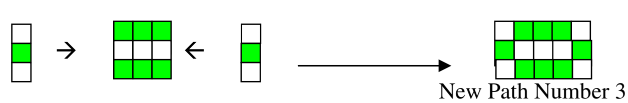


Fig. 13 Urvashi's design for new path 3 (Burrill, Cole, & Pligge, 2006, p. 3)

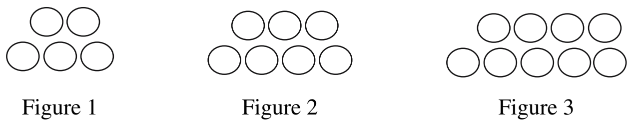


Fig. 14 Two layer circles pattern

Che: n times 2 and then you plus 2.
FDR: It's $n \times 2 + 2$. What about if I express it as n plus?
Deb: n plus n plus 2.
FDR: $n + n + 2$. Are they the same?
Jack: Yes.
FDR: Why?
Nora: You have two grey ones.
FDR: Yes, you have the two gray ones plus the two on both sides. So now if I know these formulas here, can I figure out new path number 50,000?
Students: Yeah.
FDR: So how do we do this, using the formula here. Number of whites. n plus 4 for whites. What do we do?
Tamara: It's 50,004.
FDR: What about the grey ones?
Mark: 100,002.

One indication of the students' individual appropriation of learnings from the above discourse involves their work on succeeding figural-based patterns which were mostly standard constructive generalizations that have been primarily established in figural terms. That is, what the students acquired from the above discussion was the use of figural generalizing in surfacing structural similarities among the available cues and, hence, visually identifying properties or relationships that remained stable and invariant over a sequence of cues. Further, they learned how to express those properties or relationships in algebraic form and the necessity of justifying the reasonableness and validity of the forms.

The *fourth* classroom practice came about when the students tackled the *Two Layer Circles Pattern* (Fig. 14). All the students initially perceived a recursive relation with the constant addition of one circle per layer. Two groups of students presented the formula $C = (n + 1) + (n + 2)$, where n represents figure number and C stands for the total number of circles, which they established analogically. That is, since Fig. 1 had two and three circle rows, Fig. 2 had three and four circle rows, and so on, then figure n had to have $(n + 1)$ and $(n + 2)$ circle rows. The first author then suggested organizing the two sets of numerical values in the form of a table without making any recommendation that might have encouraged a numerical strategy. The basic purpose in introducing the table in several classroom instances was primarily to foster students' growth in their representational skills, that is, patterns could also be expressed in tabular form. In the classroom episode below, Anna shared her group's thinking with the class which eventually was taken as shared and became the fourth classroom practice, that of generalizing numerically using differencing, which was reflective of an appropriation of a standard institutional numerical strategy.

Anna: We made up a formula. Like we got the figures until figure 5, and we tried it with other ones. We got $n \times 2 + 3$, where n is the figure number and timesed it by 2. So 5×2 equals 10, plus 3, that's 13. So for figure 25, it's 53.
FDR: I like that formula. So tell me more. So your formula is?
Anna: $n \times 2 + 3$.
FDR: So how did you figure this out?
Anna: First we were like making the numbers to 25. We kept adding 2 and for figure 25, it was 53.
FDR: Wait. So you kept adding all the way to 25?
Anna: Yeah… Then we used our chart. Then finally we figured out that if we timesed by 2 the figures and plus 3, that would give us the answer.
FDR: Does that make sense? [*Students nodded in agreement.*] So what Anna was suggesting was that if you look at the chart here, Anna was suggesting that you multiply the figure number by 2, say, what's 1×2?
Tamara: 2.
FDR: 2. And then how did you [*referring to Anna's group*] figure out the 3 here?
Anna: Because we also timesed it with figure number 13.
FDR: What did you have for figure 13?
Anna: That was 29. And then 13×2 equals 26 plus 3.
FDR: Alright, does that work? So what they were actually doing is this. They noticed that if you look at the table, it's always adding by 2. You see this? [*Students nodded.*] They were suggesting that if you multiply this number here [*referring to the common difference 2* by figure number, say figure number 1, what's 1×2?
Students: 2.
FDR: Now what do you need to get to 5? What more do you need to get to 5? [*Some students said "3" while others said "4."*] Is it 4 or 3?
Students: 3.
FDR: It's 3 more. So what is 1×2?
Students: 2.
FDR: Plus 3?
Students: 5. [The class tested the formula when n = 2, 3, and 25.]

5.2 An additional classroom practice in Year 2

In Fall 2006, the students were once again involved in a teaching experiment that focused on linear patterning. While the first author observed that the students, in seventh grade, seemed to have remembered how to generalize patterns figurally (weak) and numerically (strong), results of our clinical interviews with a subgroup of ten students prior to the teaching experiment confirmed this observation. In the classroom episode below, the students were asked to obtain an algebraic generalization for increasing and decreasing linear patterns in both figural and numerical forms. Emma and her group (with Drake below as a member) have been consistently applying the shared practice of generalizing numerically. However, Emma introduced her process of "zeroing out" in the case of decreasing linear patterns that resulted in a further refinement of the numerical generalizing process.

FDR: Alright. So I have my x and my y. [FDR sets up a table of values consisting of the following pairs: (1, 17), (2, 14), (3, 11), (4, 8), (5, 5), (6, 2).] So what's the answer to this one?
Drake: $y = -3x + 20$. [FDR writes the formula on the board.]
FDR: This is always the problem, here [*pointing to the constant* 20]. Before we figure that out, how did you figure out the –3?
Drake: The difference between the *y*s, between the numbers.
FDR: So what's happening here [*referring to the dependent terms*]. Is this increasing by 3 or decreasing by 3?
Students: Decreasing by 3.
FDR: So if it's decreasing by 3, what's our notation?
Students: Negative.
FDR: Alright, so negative 3. So this one is clear [*referring to the slope*]. Look at this. This one I get [*the slope*]. If you keep doing that [*i.e., differencing*], it's always true. That's why you have this. The difficult part is this [*referring to the constant* 20].

Emma raised her hand and argued as follows:

Emma: If you did a negative times a positive, it's gonna be a negative. So what I'd do is zero it out.
FDR: So what do you mean by zero out?
Emma: So like if it's –3 times 1, that's –3 [referring to the product of the common difference (–3) and the first independent term (1)]. … So I'd zero out by adding 3.
FDR: So you try to zero out by adding 3. So, what does that mean?
Emma: Coz a –3 plus 3 equals 0.
FDR: So what's the purpose of zeroing out?
Emma: So it's easier to add to 17. Coz if it's 0, all you have to do is add 17.
FDR: So you're suggesting if you're adding 3 here, if this is –3 plus 3, that goes 0. So what do you do with the plus 3 here?
Emma: Just remember it and write it down.
FDR: Suppose I remember it, adding 3. So how does that help me?
Emma: Then ahm it's easier to add to 17. So just add 17 [*to* 3 *to get* 20].

The class then tried Emma's method in a different example. The first author asked the class to first generate a table of values, and they came up with the following (x, y) pairs: (1, 10), (2, 8), (3, 6), (4, 4), (5, 2). Using Emma's method, one student offered the general formula $y = -2x + 12$, where the constant 12 was obtained after initially adding the common difference and its opposite to get 0 (i.e., $-2 + 2 = 0$) and then adding 2 to the first dependent term to yield the constant value of 12 (i.e., $2 + 10 = 12$). The class then verified that the formula worked in any instance of the sequence. Finally, when the first author asked if there was a limitation to Emma's strategy, Emma quickly pointed out that "it only works for 1" (i.e., when the case of $n = 1$ is known) and that her method would fail when the initial independent term was any other number besides 1. Hence, the *fifth* mathematical practice that became taken-as-shared was generalizing numerically using Emma's zeroing out strategy that was a further refinement of an institutional practice.

Thus in the first 2 years of teaching experiments, five fundamental mathematical practices were developed by the class as part of the process of generalization: the use of arrow strings to organize arithmetical operations; the connection between arrow strings and formulas as a means of describing invariance; figural generalization to arrive at a direct expression for a pattern; numerical generalization to arrive at a direct expression for a pattern; and the zeroing out strategy to find the value of the y-intercept of a linear pattern.

6 Middle school students' capability in justifying constructive generalizations

In various patterning studies that we have conducted with several different cohorts of learners, we saw their justification of a proposed generalization to be equally as important as their statement of generalization. Justification could mean many things (cf. Lannin, 2005), and considering the cognitive level of middle school students who are still in the beginning phase of learning domain-specific knowledge and practices in algebra, we more or less confined the notion of justification to their capacity to reason, "to understand, to explain, and to convince" (Hershkowitz, 1998, p. 29). Also, Lannin has pointed out the need to view justification in the context of generalization as "provid(ing) a window for viewing the degree to which they see the

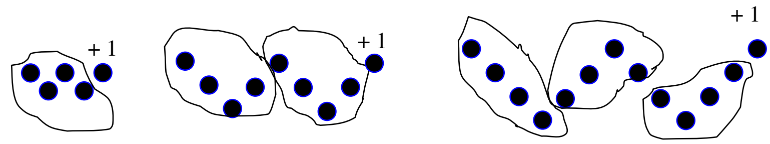


Fig. 15 Anna's figural justification of the W-dot pattern in Fig. 2

broad nature of their generalizations and their view of what they deem as a socially accepted justification" (p. 232).

Results of the Year 1 teaching experiment we implemented when our students were in sixth grade indicate differing levels of competence in the use of inductive forms of justifications. In particular, based on a follow-up clinical interview that we conducted with a group of 12 students in the class immediately after the closure of the Year 1 teaching experiment, we found that students justified in several ways, as follows: (1) all of them employed *extension generation*, that is, they used more examples to verify the correctness of their rules; (2) some used a *generic case* to show the perceived structural similarity; (3) some employed *formula projection*, a type of figural-based reasoning in which they demonstrated the validity of their formulas as they see them on the given figures, and; (4) some used *formula appearance match*, a type of numerical-based reasoning in which they merely fit the formula onto the generated table of values that they had drawn from the figural cues (Becker & Rivera, 2007). Lannin's (2005) work with his 25 sixth-grade participants used variations of strategies (1) and (2). We note that in our study, because the students in sixth grade initially developed the emergent practice of generalizing figurally, they were in fact constructing and validating their direct formulas at the same time. For example, Dung established and justified his direct expression $n + 4$ for the total number of white tiles in Fig. 13 as soon as he saw "the number of white [square tiles] in the middle plus [the] 4 white [tiles] on the sides." Also, Che, Deb, and Nora established and justified their direct expressions, $n \times 2 + 2$ when they perceived "two grey [rows] plus the two squares on both sides [in a given figural cue]." All four students came up with their inductive justifications above after empirically verifying them on several extensions and then either employing formula projection or imagining a generic case that highlights the invariant properties common to all cues. The formula appearance match was used only later after the class developed the emergent practice of generalizing numerically.

When the students in our study fully appropriated the above numerical strategies in establishing constructive generalizations, as exemplified in the thinking of Anna and Emma, we observed a shift from a figural to a numerical mode of generalizing among them. In fact, in both the pre- and post-clinical interviews in Year 2 of our study with the same group of eight students who were interviewed in the previous year, very few of them initiated a figural approach and instead most preferred to develop a generalization numerically. Consequently, such a shift affected their capacity to justify algebraic generalizations correctly on the basis of faulty responses that used either formula projection or formula appearance match. For example, Dung, in two clinical interviews when he was in sixth grade, primarily established and justified his generalizations figurally, often with the use of a generic example. However, in two clinical interviews when he was in seventh grade, Dung primarily established his generalizations numerically and justified inconsistently using formula projection. An example of a faulty argument that uses formula appearance match is exemplified in the thinking of Anna who first developed the generalization $D = n \times 4 + 1$ numerically for the pattern in Fig. 2. When she was then asked to justify her formula, she circled one group of four circles, two groups of four circles, and three groups of four circles in patterns 1, 2, and 3, respectively, beginning on the left and then referred to the last circle as the y-intercept (Fig. 15). As a matter of fact, in the post-interview in Year 2, only three of the eight students saw the sequence in Fig. 2 in the manner Dung perceived it (Fig. 3).

The phenomenological shift from the figural to numerical modes in establishing generalizations involving figural-based linear patterns among our middle school participants is not uncommon in empirical accounts of cognitive development. Induction studies in developmental psychology have demonstrated shifts in children's abilities to categorize (from perceptual to conceptual; from object- or attribute-oriented to relation-oriented, etc.). Also, Davydov (1990) has noted similar occurrences of change on the basis of his work on generalization with Soviet children, including his critique of mathematics instruction that seems to favor one process over the other. Based on the empirical data that we have collected over the course of 2 years in our longitudinal study, the shift from the figural to the numerical could be explained initially in terms of the predictive and methodical nature of the established numerical strategies. That is, the students found them to be compact and easy to use particularly in far generalization tasks in which they were asked to determine an output value for a large input value. Of course, the same characterization for numerical generalizations could also be claimed in the case of figural generalizing. However, what the students actually found difficult with the figural, which could be avoided with the numerical, was the "cognitive perceptual distancing" that was necessary to: discursively apprehend and capture invariance; selectively attend to aspects of sameness and differences among cues; and create a figural schema or a mental image of a consistent proto-cue and then transform the schema or image to symbolic terms. In terms of Radford's (2006) definition of pattern generalization—grasping of a commonality, applying the commonality to all the terms in the pattern, and providing a direct expression for the pattern—the almost, albeit not fully, automatic process of numerical generalizing requires only a surface grasp of a commonality (i.e., a common difference in the case of a linear pattern) which would then be used to set up a direct expression. In particular, when the students surfaced a commonality among cues in a numerical generalizing process involving linear patterns, most of them did not even establish it figurally since the corresponding numerical representation was sufficient for their purpose. We should also note the influence of the "whole-number bias" (following Gelman and colleagues) in cases when the students established their generalizations numerically, that is, they found the numerical approach was easier to use in patterning tasks that were increasing rather than decreasing.

In articulating our argument of a shift in mode of generalizing that took place among the middle school children in our study, we have already noted how most of them could correctly establish constructive generalizations numerically but had difficulty justifying them. Further, we already discussed how some of them employed formula projection in an inconsistent (faulty) manner. Another significant source of difficulty in justifying was the students' misconstrual of the multiplicative term in the general form $y = mx + b$ for linear patterns. Towards the end of the Year 1 teaching experiment, they would often express their algebraic generalization in the form $O = n \times d + a$, where the placeholder O refers to the total number of objects being dealt with (like matchsticks, circles, squares, etc.), n the pattern number, d the common difference, and a the adjusted value. For example, the general form for the pattern sequence in Fig. 1 is $T = n \times 3 + 1$. The students would then justify the form by locating n groups of three matchsticks respecting invariance along the way. In the Year 2 study, they learned more about the commutative property and then wrote all their generalizations in the equivalent form $O = dn + a$. However, they got confused because they interpreted the expressions $n \times d$ and $d \times n$ as referring to the same grouping of objects. For example, in the clinical interviews that we conducted immediately after the Year 2 teaching experiment, some of those who wrote the form $D = 4n + 1$ for the sequence in Fig. 2 justified its validity by looking for four groups of, say, two circles in pattern 2 when, in fact, they should have been looking for two groups of four circles. Thus, the algebraic representation proved to be especially difficult among those who established their generalizations numerically because of misinterpretations involving some of the mathematical concepts and properties relevant to integers (such as the commutative law for multiplication).

7 Middle school students' capability in constructing and justifying deconstructive generalizations

Considering the results drawn from our longitudinal work and relevant patterning studies discussed in this paper, we can conclude with sufficient sample that the task of establishing and justifying a deconstructive generalization is difficult for most middle school children. Why it is so remains an unresolved issue. We do not know the weight, much less the content, of the contributing factors that influence students' capacity for deconstructive generalizing. Further, we remain unsure whether such factors are developmental-sensitive, learning-driven, or something else. It is certainly plausible to think that, from a rational constructivist perspective, developing an operational schema that is appropriate in a deconstructive generalization could not be accomplished easily since both figural and numerical requirements do not align or fit with the existing core domain-specific structures of middle school children.

Also, we considered the possibility that students with a predominantly figural predilection to see patterns might be more likely to succeed in deconstruction tasks than those who generalize in a predominantly numerical mode. For example, Emma, in a clinical interview before the Year 2 teaching experiment took place, initially employed a figural approach in dealing with the pattern in Fig. 1. She first counted the squares for patterns 1, 2, and 3 and then built pattern 5 with the toothpicks. After she had counted 16 toothpicks for pattern 5, she then reasoned as follows: "[Pattern 5] has 16 because $4 \times 5 = 20$ and since you had 16 before, you have extra ones in there, so subtract 4 [and] you get 16." In establishing a formula, she reasoned analogously in the following manner: "$P = (n \times 4) - 4$ to get to 16. Are we always going to take away 4? Look at pattern 3. $4 \times 3 = 12$, so subtract from pattern number, how do you say that? One less than n." She used the same figural scheme to calculate the required number of toothpicks in pattern 15. However, when Emma started feeling overwhelmed with having had to account for two constraints in symbolic form [i.e., $(n \times 4)$ for the total number of toothpicks and $(n - 1)$ for the number of overlaps that need to be taken away from the total], she gave up her figural strategy since it became complicated for her. She then resorted to establishing a constructive numerical generalization (i.e., $T = 3n + 1$). Hence, figurally establishing and justifying several different parts and then expressing them as a deconstructive unit can be a difficult process for many middle school children.

It could also be the case that deconstructive generalizing depends on the nature and complexity of a patterning task, including the instructional mediation used in encouraging students to think in deconstructive terms. Results of the two clinical interviews in our Year 2 study separated by a teaching experiment on deconstructive generalizing show the students had more difficulty dealing with the W-dot pattern in Fig. 2 than the adjacent squares pattern in Fig. 1. In particular, results of the clinical interview with ten students prior to the teaching experiment show only one student correctly justified a deconstructive formula in the case of the W-dot pattern and six students in the case of the adjacent squares pattern. Further, all eight students interviewed after the teaching experiment were able to justify the deconstructive formula for the square toothpicks pattern, but only six students in the case of the W-dot pattern. Thus, it seems that some overlaps in a deconstructive generalization task are easier to see than others. For example, the students above found it easier to see overlaps among the shared adjacent sides of the squares than the shared interior vertices in a W-dot formation.

Finally, even in the context of a teaching experiment in which middle school children are provided with an opportunity to acquire experiences relevant to deconstructive generalizing, deconstruction continues to be a difficult task. Steele and Johanning (2004) developed a teaching experiment in which eight US 7th graders were asked to generalize five linear and three quadratic problem situations that pertained to growth, change, size, and shape. Their results show that, in the case of tasks that contained figural cues, only three students established and justified deconstructive generalizations (or "well-connected subtracting-out schemas"). In the clinical interviews with ten children that we conducted in Year 1 after the teaching experiment took place, no student was found capable of establishing and justifying a deconstructive generalization. Further, in clinical interviews with eight children in Year 2 after a teaching experiment, none of them were still capable of constructing such forms. However, there was a significant improvement in their ability to interpret and justify a stated deconstructive generalization. All eight students saw the overlapping sides in the adjacent squares pattern in Fig. 1, and six could see the overlapping interior vertices in the case of the W-dot pattern in Fig. 2. We further note that despite their success in justifying, seeing an overlap was not immediate for most of the students; it became evident only after they had initially employed formula appearance match followed by formula projection. Of course, some students employed formula projection incorrectly. For example, Jana justified the subtractive term 3 in Zaccheus's deconstructive generalization (item D in Fig. 2) in the following manner:

FDR: So if you look at this [*referring to the formula* (*item D, Fig.* 2) *in which Jana substituted the value of 2 for n*], this one's four times two plus one, right? And then minus 3. So how might he be looking at 4 times 2 plus 1 and then minus 3?
Jana: Uhum, the 2 is for the pattern number.
FDR: Uhum. Because when Zaccheus was thinking about it, he said multiply 4 by $n + 1$ and then take away 3. So how might he be thinking about it?
Jana: Like it's gonna be 3 [*referring to* 2 + 1] and then it's gonna be 12 [*referring to* 4×3]. But I counted there's only 9, so he has to subtract 3.
FDR: So how might he be doing that? Suppose I do this? [FDR builds pattern 2 with circle chips in which the three overlapping "interior" vertices are colored differently.]
Jana: Hmm, like he has this group of 4 [Jana sees only two sides in W in pattern 2 with the top middle interior dot connecting the two sides. Hence, one side has four dots.].
FDR: Is there a way to see these 4 groups of 3 here [*referring to pattern* 2]?
Jana: Like he imagines there's 3 and he has to subtract 3.
FDR: So can you try it for other patterns? [*Jana builds pattern* 4.]
Jana: He has 1 group of 4. So there's 3 groups of 4 and he imagines 3 more [*to form* 4 *groups of* 4] and then he subtracts them [*the three circles added*].
FDR: So he imagines there's three more. But why do you think he would add and then take away?
Jana: Because there's supposed to be 4 groups of 4 and then you don't have enough of these ones [circles] so he adds 3. You add these ones.

8 Conclusion

This paper began with two broad questions that have guided the longitudinal research program summarized in this work: what is the nature of the content and structure of generalization involving figural patterns of middle school learners? And to what extent are such learners capable of establishing and/or justifying more complicated generalizations? Various patterning studies that have been conducted at the middle grades level provide strong evidence that students' generalizations shift from the recursive to the closed, constructive form. In this article, we discussed in some detail at least three epistemological forms of generalization involving figural-based linear patterns, namely: constructive standard; constructive nonstandard; and deconstructive. The general forms are further classified according to strategy complexity, with constructive standard as being the easiest for most middle school children to establish and, thus, most prevalent, constructive nonstandard as being slightly difficult, and deconstructive as the most difficult to achieve. This classification scheme emerged from detailed analyses of students' attempts at generalization over two full academic years, and elucidates the content and structure of such generalizations.

We have also discussed how students' approaches to establishing generalizations are intertwined with their justification schemes. Results drawn from our longitudinal work show middle school students' cognitive tendency to shift from a figural to a numerical strategy in establishing figural-based patterns. We note two consequences. First, we note changes in their representational skills and fluency, that is, from being verbal (situated) to symbolic (formal). Second, such a phenomenological shift affects the manner in which they justify their generalizations. We have documented at least four types of inductive justifications, namely: extension generation; generic example use; formula projection; and, formula appearance match. The entry level of inductive justification often involves generating extension cues. Students who then generalize numerically without having a strong figural foundation are most likely to employ formula appearance match and use formula projection inconsistently.

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