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ORIGINAL ARTICLE

From math drawings to algorithms: emergence of whole number operations in children

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Abstract This paper provides a longitudinal account of the emergence of whole number operations in second- and third-grade students (ages 7–8 years) from the initial visual processing phase to the converted final phase in numeric form. Results of a notational documentation and analysis drawn from a series of classroom teaching experiments implemented over the course of two consecutive school years indicate successful conceptual progressions in participants' epistemological and synoptical use of visual representations. Further, their developing symbolic competence in whole number operations underwent several phases from initially linking notations with their respective signifieds to developing, elaborating, and routinizing symbol manipulation rules. Progressive emergence in both use and competence operated within interacting cycles of abduction, induction, and deduction.

1 Introduction

This paper provides a longitudinal account of the emergence of whole number operations in children ages 7–8 years from the initial visual processing phase to the converted final phase in numeric form. Such exact understanding of operations often emerges in mathematical activity with relevant intentional tools [e.g., Dienes blocks and math drawings (Fuson 2009)] and in learning contexts and mathematical practices (Font et al. 2013) that support growth in necessary mathematical knowledge. In this paper, the following research question is addressed in the context of a series of classroom teaching experiments implemented over the course of two consecutive school years: *How do second- and third-grade students process and convert visual-driven representations of whole number operations in mathematical form?* The research question underscores conceptual phases they go through as a consequence of changes that occur in their understanding of whole number operations. The term "operations in mathematical form" underscores the significance of deductive closure in analytical processing of visual representations. That is, when children manipulate such visual forms, meaningful inference and thinking should enable them to abduce rules or principles (i.e., produce conjectures), verify them through induction, and deduce generalizable rules or principles that they can apply to novel and other complex situations. Inference means "going beyond data," while thinking involves "deliberately applying and coordinating [the] inference to fulfill a purpose" (Moshman 2004, p. 223).

In this study, visual strategies play a mediating role in the emergence of children's sophisticated, structured, and necessary understandings of mathematical objects. Such strategies provide "imaginal support" that enable them to be creative, experience breakthroughs, and develop "self-evidence" of, including "empathy" toward, the intended mathematical relationships (Fischbein 1987; Rivera 2011). Children also use visual strategies to help them conduct explorations, organize relevant data, and anticipate an intended analysis. Further, such strategies enable them to develop "a feeling of intrinsic certainty" and "sensori-mental structure" (Fischbein 1987) and an awareness of the need to establish rules and reasons (Duval 2002). In fact, the visual forms they generate convey the manner in which they justify in a practical context, where the relevant visual manipulations reflect an emerging structure about ways in

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which they conceptualize such objects (e.g., Csikos et al. 2012; Martin et al. 2007; Rivera 2013).

The data reported in this study have been drawn from a 2-year longitudinal project with an elementary class of 21 students in an urban classroom in Northern California, USA. The students (7 girls, 14 boys; 20 Hispanics and 1 African-American) were in second grade when they started the project. Only ten of them met first-grade level state standards in mathematics on the basis of a districtwide summative test that was administered in the preceding school year. Together with their officially designated teachers (two teachers in second grade and one teacher in third grade), the author designed and implemented sequences of teaching experiments that engaged the students in a progressive visualization of mathematical ideas. Funding for the longitudinal project was premised on the conditions of broadening the participation of all groups and conducting interventions in schools and classrooms that needed them the most. The urban school that participated in the study, with a profile of about 95 % minority students and about 81 % English learners, consistently had to deal with second grade students who could not achieve proficiency and advanced levels (i.e., meeting and exceeding grade-level standards, respectively) in the state mathematics assessment. The school's second-grade score averages in mathematics in the years prior to the study were below proficient, ranging from 35 to 61 %.

2 Theoretical framework

The theoretical framework for the study embraces Duval's (2002) semiotic representational perspective of mathematical objects—that is, such objects cannot be directly accessed because they are primarily ideas. Consequently, learners can only operate on their representations that make them perceivable, observable, and intelligible in an appropriate semiotic format. Hence, when they visualize a mathematical object or process, it means that they "see" its essence as it is grounded in some representational context. Visualization then becomes representational and focused, which makes it conceptually different from vision that sees objects in complete wholes.

Visualization is, thus, loaded with *epistemological* and *synoptic functions.* When signifieds undergo visual representing, learners activate an epistemological function in which case signifiers are assigned to signifieds. Visualization also activates a synoptic function when learners analytically see through the objects and processes via focusing actions. In vision, learners see what they see and experience and often "require (physical) movement" in order to achieve "complete apprehension" (Duval 2002, pp. 320–322). In visualization, however, they see "neither [a] mental nor physical" object but its "semiotic" form. This way of seeing enables them to construct and deduce *discursive representations* that "can get at once a complete apprehension" or a "synoptic grasp" of "relations" and "organization of relations between representational units" (ibid.).

While discursive representations apply to intentional images (see Fig. 1), individual learners' mathematical "cognitive architecture" draws on different kinds of representations. Thus, the primary mathematical learning dilemma involves knowing how to effectively coordinate between inter- and intra-representational issues in mathematical knowledge construction and understanding (cf. Tsamir 2001; Bagni 2006; Gal and Linchevksi 2010; Godino et al. 2011; Santi 2011; Rivera 2013). *Intra-*

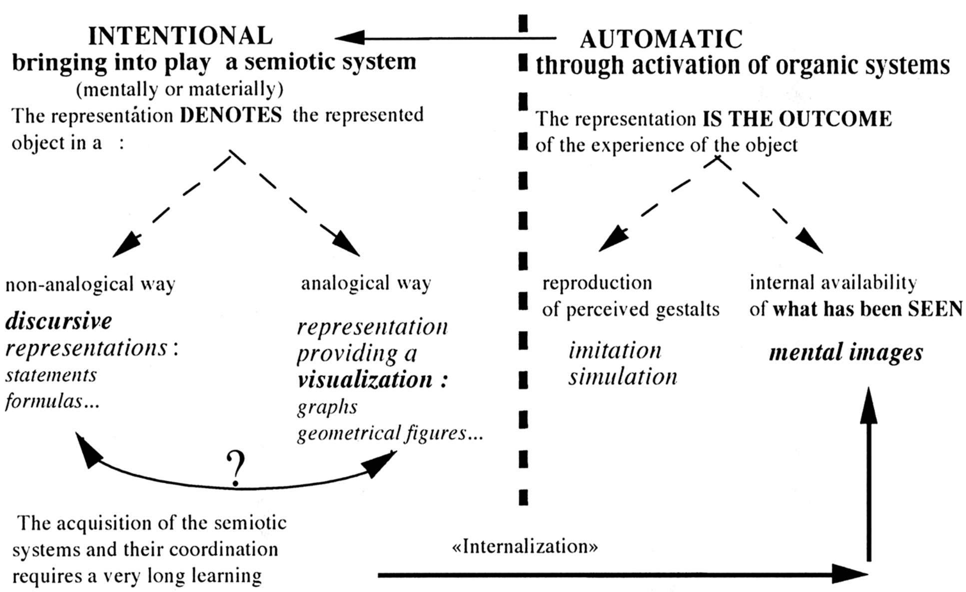

Fig. 1 Classification of image-based (conscious) representations (Duval 2006, p. 315)

representational issues involve *transformational processing* within the same representational medium (e.g., counting-all to counting-on when adding two whole numbers). *Inter-representational issues* involve *transformational conversion*(*s*) between two or more representational systems. For example, figurate number patterns can be expressed visually, numerically, algebraically, and graphically, conveying different but equivalent systems. For Duval (2002), *conversion* "is a crucial problem in the learning of mathematics"—that is, "if most students can learn some [transformational] processing, very few of them can really convert representations. Much misunderstanding stems from this inability" (p. 318; cf. Duval 2006). Conversion is also confounded by a translation dilemma—that is, either two representations may not look congruent (i.e., they are not transparent) or their congruence may not be bidirectional.

Figure 1 also underscores the "bringing into play a semiotic system" in intentional images. Hiebert's (1988) theory of developing competence with written mathematical symbols is a useful and complementary way of operationalizing such a system. The theory involves the processes below. The first two processes are the meaning-making phases, where notations and actions are linked in some familiar quantitative context in which they emerge. The remaining three processes shift the "concern with meaning to a concern with power and efficiency [and transfer]" (ibid., p. 341).

(1) Connecting individual symbols with referents;

(2) Connecting individual symbols with referents;

(3a) Elaborating procedures for symbols;

(3b) Elaborating procedures for symbols;

(4) Using the symbols and rules as references for building more abstract symbol systems. (Hiebert, 1988, p. 335)

Students in the *connecting phase* of a semiotic system first establish a correspondence between written notations and quantities (e.g. whole numbers) and actions (e.g. addition) on the quantities. Meaningful connections yield transparent notations that display properties of their specific referents. Such notations remain at the level of concrete representations and "provide mental paths" to their respective referents. Also, the relevant actions convey anticipated actions that do not necessarily include "knowledge of the algorithms used to generate answers" (ibid., p. 337).

In the *developing phase*, they formulate procedures by manipulating the referents and "paralleling the action" (ibid., p. 338) on either notations or actions. Further, such procedures are considered successful and valid if the notations and actions "faithfully reflect" (ibid., p. 339) the manner in which they are manipulated at the level of referents.

In the *elaborating phase*, they extend the same procedures to novel and complex tasks that cause them to reflect on the procedures (ibid., p. 341). Hence, meanings shift from their concrete referents to the "rule system itself" (ibid., p. 343) that supports repeated actions and pays attention to the relationships between notations and actions on the notations. Such procedures are successful if they are consistent, that is, they can be applied to all familiar, relevant, and equivalent contexts.

In the *routinizing phase*, they execute procedures with minimal effort, seeing them as routines that "facilitate further understanding of the system" (ibid., p. 344). In the final phase, *building more abstract notational systems*, they learn to "transfer meaning from a familiar [notational] system to a new, more abstract system" (ibid., p. 344).

3 Elementary students' understanding of whole number operations: a brief review

The development of teaching experiments for this particular study drew on the research syntheses of Clements and Sarama (2009) and Verschaffel et al. (2007) regarding young children's understanding of whole number concepts and operations. Their syntheses have overlapping features as a consequence of converging evidence from the field. In this brief review, while only their work is referenced, the underlying basis for such a reference, in fact, draws on this extensive body of research. A good starting point involves the ability called *conceptual subitizing*, which enables children and adults to quickly see and combine parts to make wholes. Four-to-five-year-old children initially learn to perceptually subitize four to five objects (i.e., judge their cardinality quickly without counting them one by one). At ages 5 and 6, they learn to conceptually subitize to 20. From age 7 to 8, they are able to conceptually subitize with place value, skip counting, and multiplication. Progressions in the use of counting principles that apply to single-digit whole number addition and subtraction situations tend to proceed as follows:

- counting all with concrete objects;
- counting all without concrete objects;
- counting on from the first number;
- counting on from the larger number;
- using derived combination strategies (e.g., knowing doubles statements);
- and mastery (i.e., automatic retrieval) of the different arithmetical combinations.

In the case of multiplication, progression proceeds in the following manner:

- counting all with concrete objects through repeated addition and additive doubling;
- using patterning strategies (e.g. multiplying by 7 involves adding multiples of 3 and 4) and other derived combination techniques;
- and mastery of the multiplication table of products.

More complicated cases of place-value operations involving two or more digits of whole numbers rely on a firm grasp of the decimal number system. Additionally, Verschaffel et al. underscore macrocultural conditions (e.g., family and culture) and recent classroom restructuring initiatives that encourage young children to use mental arithmetic in the case of more general structures of whole number operations.

The five phases below are based on the work of Fuson and colleagues (e.g., Fuson et al., 1997). They describe elementary school children's conceptual progression in their place value understanding of whole numbers. The phases are not permanent due to factors such as language use, developmental constraints, conceptual supports within and outside the classroom, and students' preference to use different strategies depending on what they find meaningful in an arithmetical situation. Also, the phases have implications in how children perceive different representational relationships that also depend on their familiarity with different sets of numbers.

[Phase 1; *Unitary multidigit conception*] Whole-number quantities are seen as individual units with no sense of grouping structure in, say, a place value context.

Example: A set of 12 objects is not interpreted in a place value context and there is no evident grouping by tens and ones.

[Phase 2; *Decade-and-ones conception*] Numerals, say 2-digit ones, are slowly being separated into two quantities with units, which convey a beginning understanding of place value structure.

Example: The numeral 12 is seen as consisting of 10 objects and 2 objects.

[Phase 3; *Sequence-tens-and-ones conception*] Whole-number quantities at this stage are constructed in terms of sequences of units in a place value structure.

Example: A set of 12 objects is counted in groups of ten, which also means inferring a decades-structure in the process of counting.

[Phase 4; *Separate-tens-and-ones conception*] Whole-number quantities and their numeral representation are interpreted as conveying the union of separate units.

Example: The numeral 12 is seen as consisting of two separate units, that is, a unit of 10 and units of 1. Also, a unit of 10 is flexibly seen as consisting of 10 ones.

[Phase 5; *Integrated sequence-separate tens conceptions*] Whole numbers are seen as conveying both sequence (level 3) and separate (level 4) concepts in a bidirectional context. Representational fluency in the use of number words, marks, and quantities for the same whole number is also evident. Students in this phase exhibit flexibility that enables them to generate different approaches and strategies for solving problems involving, say, 2-digit numbers. (cf. Clements and Sarama 2009; Verschaffel et al. 2007)

Once elementary students become proficient in phases 3 and 4, they learn to operate on two or more whole numbers in a place value context. In addition and subtraction situations, for instance, those who employ a *splitting* strategy partition all the numbers by place value, focus on the digits, count and regroup them when it becomes necessary, and then record the final answer following the standard notation. Some employ *jumping* (or *chunking* or *positional*) in which case they start with an unsplit whole number and then in a sequential manner count up or down to the next number by place value chunking until they obtain the total value. From a multiplicative thinking standpoint, it is only in Phases 3 and 4 when students can reason explicitly about numbers in units and visualize them as both whole sets and composites of units. In informal sense-making contexts, however, they have been documented to produce mental strategies and variations of splitting and jumping that do not strictly draw on place value but reflect what they consider to be meaningful and familiar mathematical relationships among the numbers. Some well-known strategies involve compensation, general decomposition, purposeful jumping, and combinations of such strategies. Students also use similar strategies in the case of multiplication and division of whole numbers.

This particular study utilizes the above framework of five phases in describing the participants' progressive conceptual understanding of place-value splitting operations involving whole numbers. However, such descriptions begin in Phase 2 when students begin to think of whole numbers and their operations in terms of structures. To illustrate, Table 2 shows the conceptual phases for adding and subtracting whole numbers up to two digits among second-grade students in a splitting context. The initial phase corresponds to Phase 2, which supports place-value manipulations. Within each phase, the following aspects are also described: (1) the content of processing and conversion exhibited by the students; (2) the quality of their mathematical inferences; and (3) either the

Table 1 Second grade student responses on the tasks shown in Fig. 1

Task	Count all with concrete or physical models or pictures	Count on from the first number	Count on from the larger (or smaller) number	Place value splitting	Derived combination strategies	Mastery of arithmetical combinations	Incorrect responses
1	10	0	1	2	3	0	5
2	13	0	3	0	0	0	5
3[a]	8	6	0	0	0	0	5

[a] 19 students

epistemological or synoptical functions of the relevant visual representations.

In this paper, the descriptions of changes basically draw on an analysis of notations that the students exhibited in purposeful mathematical activity for two related reasons. First, elementary students have been documented to be capable of inventing, appropriating, and transforming conventional notations especially in classroom teaching experiment contexts that are designed to support such actions (cf. Brizuela 2004). Second, inferences that accompany their use of such notations are often implied and they are unaware of them. That is, as Moshman (2004) points out based on studies with young children, "what develops beyond early childhood is not the basic ability to make logical inferences, but metalogical knowledge about the nature and justifiability of logical inferences, and metacognitive awareness, knowledge, and control of one's inferential processes (ibid., pp. 222–223).

4 Methodology

4.1 Official content on whole numbers and operations from first through third grade

First-grade students (mean age of 6 years) in California learn about whole numbers up to 100 by using physical models and diagrams, forming number expressions, and counting and grouping objects in ones and tens with the aid of Dienes blocks. Conceptual subitizing opportunities occur through repeating and linear patterning activities and skip counting by 2, 5, and 10 s to 100 that emphasize "unit" understanding in both geometric and numerical contexts. They also learn to add and subtract whole numbers up to 100, obtain the sum of three one-digit numbers, and create, write, and solve arithmetical problems involving addition and subtraction. Regrouping is limited to counting on from the first number, counting on from the larger number, and counting on from the smaller number. In second grade, they deal with numbers up to 1,000 and learn to formally add and subtract whole numbers with regrouping. Multiplication is formally introduced using repeated addition and rectangular arrays and is reinforced through skip counting activity by 2, 5, and 10 s. Division is formally introduced through repeated subtraction, rectangular arrays, equal sharing, and forming equal groups with and without remainders. In third grade, they work with the base-10 structure of whole numbers and operations up to 10,000 and add, subtract, multiply, and divide whole numbers in their standard formats. Products in multiplication do not exceed 10,000. Further they learn to commit to memory the multiplication table with factors up to 10. Division is still limited to single-digit divisors and quotients that do not exceed 10,000 (California Department of Education 1999).

4.2 Development of the 2-year teaching experiments

An initial formative assessment consisting of the three tasks shown in Fig. 2 was conducted during the first few weeks of the school year. Results of the informal data analysis were used to design the first teaching experiment on whole number concepts. Further, results of a district-administered benchmark assessment covering several objectives on whole number concepts became the basis for designing the second teaching experiment that focused on

Fig. 2 Three arithmetical problems given to grade 2 students in the initial assessment

1. [In-class work] The museum shop has 6 fossils. Then it buys 38 more. How many fossils does the shop have now?

2. [In-class work] The museum shop has 12 model rockets. 4 are sold. How many model rockets are left?

3. [Clinical interview task] What number will complete the following statement? 13 + ___ = 21. How do you know for sure? How many answers are possible?

addition and subtraction of whole numbers. A second district assessment was then administered and analyzed, which signified the closure of the second teaching experiment. The cyclic triad of external assessment (EA_i)—development of teaching experiment—external assessment (EA_{i+1}) was the underlying design structure of the 2-year sequence of classroom teaching experiments on whole number operations. The teaching experiments were implemented in sequence, following the district's pacing guide and assessments.

The development and implementation of every teaching experiment followed a protocol. In the development phase, relevant findings from research regarding a target operation and a proposed learning trajectory were discussed among the research team that consisted of the author, graduate assistants, and the teachers. The teachers' perspectives, concerns, and insights regarding students' difficulties and strengths were especially noted. Activities were then developed and negotiated. While the activities generally reflected content requirements stipulated in the state standards and district pacing guides, the author also developed the content based on the agreed trajectory and aligned with the proposed visual model for thinking about whole number operations.

In the implementation phase, the teachers often began the discussion together as a whole class, which lasted no more than 10 min of class time. Since young children enjoy listening to stories, the teachers used them quite frequently as a context for discussing and processing academic language, concepts, and processes that the students needed to know in order to accomplish the follow-up individual work, which usually took about 30 min of class time. On follow-up days, the teachers used the beginning phase of class time to praise the class for work already accomplished. This cued the students to share their strategies with the class. The teachers carefully processed the responses in sequence from the least effective to the most sophisticated. The students then used that shared knowledge to continue working on activities independently.

During the independent phase, members of the research team helped students accomplish the activities. Any issue that came up was pointed out quickly and discussed by the team. Interesting strategies and insights from the students' work were also brought up. Issues, strategies, and insights that merited a whole-class discussion meant motioning the whole class to regroup, which cued the students toward closure. However, in situations when a classroom event occurred as planned, closure often had the students engaged in other activities that helped them practice and strengthen their other mathematical skills. Follow-up emails were employed to continue the discussion, share strategies, and prepare succeeding activities with the students.

4.3 Data collection and analysis

In the first year of the study, data were collected and analyzed thematically following grounded theory protocols (Glaser 1992). For example, the author and a graduate student assistant generated frequencies based on type of strategies that emerged from the clinical interviews (see Table 1). The research team used results from students' classroom work as a basis for all follow-up activities. Benchmark summative assessments were also collected every 6 weeks, and items were analyzed for strengths and weaknesses. District staff external to the team provided such useful information that allowed the team to assess the short-term effects of all the teaching experiments. Formative assessment data that were collected during a teaching experiment phase included quizzes, homework, and classroom work; these were organized, labeled, and dated in individual folders. In the second year of the study, the primary sources of data were the benchmark and formative assessments.

In this study, changes in students' conceptual understanding of whole number operations were documented by analyzing their written work in various stages of every teaching experiment. Since the social aspect of every teaching experiment was carefully planned and controlled to mediate effectively in the development of notations and processes among the students, the analysis in this paper addressed ways in which notations became recognizable and were decoded by the students. Further, the research team was involved in open coding of all students' work that provided them with shared language to talk about students' thinking at various points in any experiment. The codes underwent naming, comparing, merging, modifying, and renaming over time. The author further engaged in selective coding of the open codes, which involved testing and validating the open codes on copies of students' work. When selective codes appeared consistent, that became the basis for formally establishing theoretical codes. Every theoretical code consists of a description and representative samples of student work. The results presented and discussed in Sect. 5 represent theoretical codes that emerged from both open and selective coding processes (Glaser 1992).

5 Results

This section describes conceptual phases in the participants' processing and conversion of visual-driven representations of whole number operations in mathematical form over the course of 2 years. The descriptions also include illustrations that show how and when they used the relevant representations in deducing relevant mathematical relationships.

5.1 Unitary visual phase for addition and subtraction

Table 1 is a summary of the second-grade students' responses on the three tasks shown in Fig. 2. Counting all with the aid of external tools was consistently the most favored operational strategy. Overall the students manifested a Phase I conception of whole numbers. At the epistemological and synoptical levels, circles, sticks, fingers, and the numerals on a number line were interpreted as individual objects that were used mainly for counting.

5.2 Conceptual processing-conversion phases for addition and subtraction in grade 2

Table 2 summarizes the underlying conceptual changes and actions that the second-grade students manifested for the operations of whole number addition and subtraction in two sequences of teaching experiments that spanned 10 weeks. Second-grade Lisa's work in Fig. 3a–c exemplifies the transformational processing and conversion that took place in her subtraction process. Figure 3a, b exemplify structured thinking at Level III, where Lisa conceptualized the numbers and the operation in both their numerical and visual formats together. For example, she initially interpreted $32 - 18 = 14$ as "*taking away 1 stick and 8 dots from 3 sticks and 2 dots,*" which she then processed as "*taking away 1 stick and 8 dots from 2 sticks and 12 dots*." Her converted numerical form was a faithful recording of her processing in visual form. Figure 3c exemplifies Lisa's integrated thinking at Level IV. In this situation, the visual representation was replaced by a deductively drawn discursive representation in numerical form. That is, her numerical solutions reflected how she conceived of subtraction discursively as operating along rules that involved regrouping and taking away. Also, the deductive component in her thinking in Fig. 3c enabled her to apply the subtraction rule in a variety of new tasks without the need to draw numbers in terms of sticks and circles.

5.3 A visual refinement in subtraction regrouping in grade 2

Figure 4b shows samples of Gerry's modified understanding of place value splitting in the case of subtraction with regrouping. Normally, students at Level III in Table 2 performed subtraction with regrouping by employing a combined decomposition–recomposition process. In the case of $41 - 16$, for example, Gerry in Fig. 4a initially decomposed a stick into 10 circles, redrew the 10 circles in the ones place, recomposed the circles in the ones place by engaging in a

Table 2 Conceptual visual-to-numerical phases for adding and subtracting whole numbers in grade 2

Conceptual phases	Transformational processing	Transformational conversion	Relevant mathematical inference/s being used	Function of visual representations
I. Binary decade-and-ones conception (e.g., Figs. 3a, 5)	Place value discernment, where the digits of whole numbers being added or subtracted are seen in terms of sticks (for tens) and dots (for ones)	Numerical to visual fluency and vice versa, no operation evident at this stage	Abductive inference of place value structure and inductive verification of structure through examples	Epistemological discernment of the relationship between math drawings and place value units
II. Binary sequence-tens-and-ones conception (e.g., Figs. 3a, 5)	Place value construction and ungroup processing, where the digits of whole numbers are added or subtracted sequentially in terms of tens and ones	Visual and numerical fluency in adding and subtracting without regrouping	Deductive implementation of math drawings and rules for adding and subtracting without regrouping	Synoptical understanding of adding and subtracting digits in the context of quantities
III. Binary separate-tens-and-ones conception (e.g., Figs 3a, b, 5)	Complex place value processing, where the digits of whole numbers being added or subtracted are ungrouped and regrouped according to their unit requirements	Visual and numerical fluency in adding and subtracting with regrouping	Deductive acquisition of rules for adding and subtracting with regrouping	Epistemological understanding of the equivalence relationship between 1 stick and 10 dots; synoptical understanding of the regrouping process
IV. Binary integrated sequence-separate conception (e.g., Fig. 3c)	Discursive representations, i.e., statements of rules for adding and subtracting numbers in ungrouping and regrouping contexts	Numerical recording of an internalized visual process for adding and subtracting numbers in horizontal and vertical formats	Deductive action on numerical tasks that involve addition and subtraction	Empirical support when appropriate

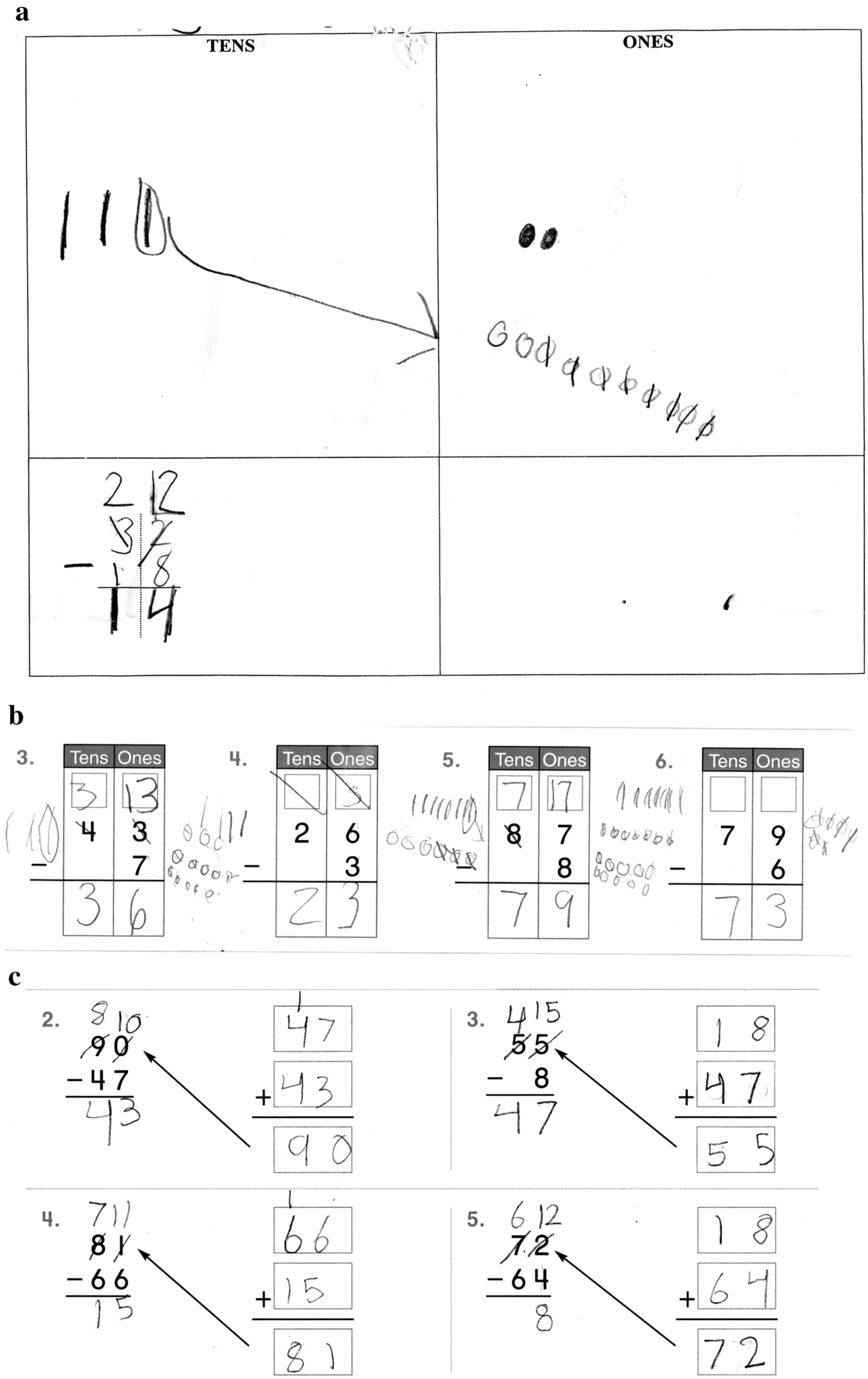

Fig. 3 Second grade Lisa's Level III to Level IV subtraction processing. **a** Visual stick-and-circle subtraction on a place value mat. **b** Visual stick-and circle subtraction on a textbook. **c** Numerical-based subtraction

counting-all strategy ("1, 2, 3, …, 10, 11, 12, 13") in order to obtain the total, which he then recorded above the ones column as "11." Most of the students, in fact, manifested this regression to counting-all action. That action unfortunately prevented them from smoothly transitioning to Level IV. The dilemma was resolved when their teachers asked them to skip the decomposition-of-a-stick action in favor of direct whole-stick regrouping as shown in Gerry's work in Fig. 4b, which facilitated counting-on action. For example, in the case of 81 − 32, Gerry knew that regrouping was needed in the ones place. So he regrouped the 8 sticks into 7 sticks and 1 stick, took away 1 stick from the set of 8 sticks and added 1 stick in the set of 1 circle, and then performed take-away subtraction. His recording of the value under the "new" ones place was made possible by his ability to engage in Level I conceptual subitizing.

5.4 Extended diagrammatic place value splitting in grade 3

In third grade, the students once again drew on their earlier visual structured experiences in dealing with whole numbers of up to five digits. Figure 5 exemplifies a student's written work on addition and subtraction problems involving larger numbers. When the students were initially presented with four-digit whole numbers, they decided to use rectangles to represent a thousand digit. They then proceeded in the usual manner. Further, it took them only one classroom session to recall the visual-numerical processing for adding and subtracting whole numbers by splitting with and without regrouping. In fact, both simple and complex de/re/composition cases were pursued together with relative ease with the ungrouping situations considered as special cases.

Results of the students' average proficiency percentage involving adding and subtracting whole numbers with and without regrouping on the district and state assessments over the course of 2 years was about the same, 82 %, which was significantly much better than their prior school averages in the previous years.

5.5 Conceptual processing-conversion phases for multiplication in grades 2 and 3

Table 3 summarizes the conceptual phases for multiplying whole numbers from single-digit factors to place-value splitting contexts that emerged in the students' activity over 2 years. Their initial formal experiences with the operation of multiplication had them generating several different addition sentences for an unordered set of 6 circles. Responses that conveyed repeated addends became the basis for processing the meaning of multiplication of two whole numbers and converting to the notation $a \times b$, where a refers to the number of equal groups and b the

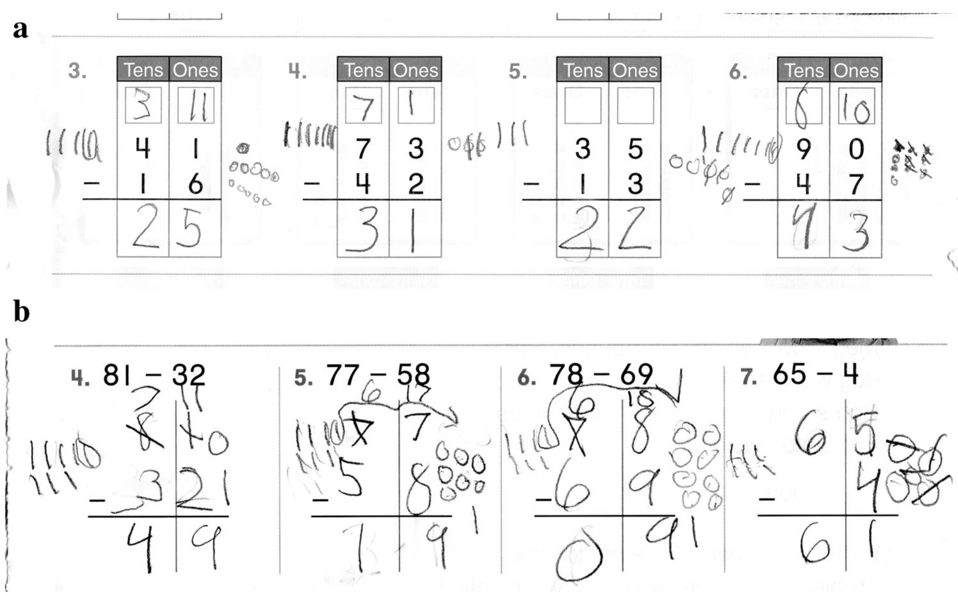

Fig. 4 Second grade Gerry's transformation processing of subtraction with regrouping action. **a** Combined decomposition–recomposition action. **b** Direct recomposition

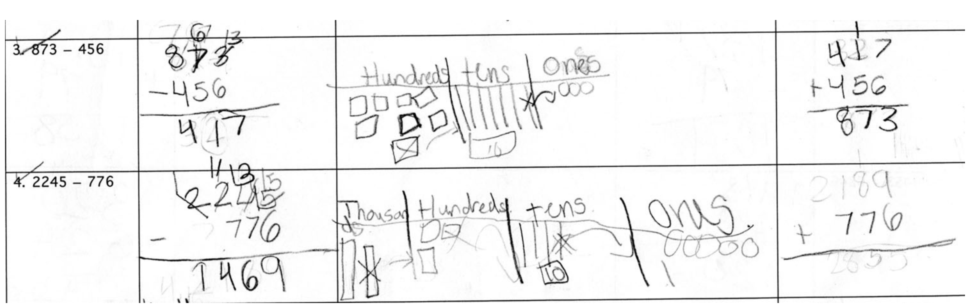

Fig. 5 Combined visual-numeric adding and subtracting in grade 3

Table 3 Conceptual visual-to-numerical phases for multiplication from grade 2 to grade 3

	Grade level	Conceptual phases	Transformational processing	Transformational conversion	Relevant mathematical inference being used	Function of visual representation
Grade 2		I. Early multiplicative visual conception: Expanded modeling (e.g., Figs. 6, 7)	Repeated unit discernment, where a given quantity (i.e. product) is seen as the union of equal groups of the same unit (multiplicand)	Visual to numerical fluency and vice versa, emergence of the symbolic notation $a \times b$	Abductive inference of two single-digit whole number multiplication as repeated addition and inductive verification through examples	Epistemological discernment of forming equal groups relative to a common unit; Synoptical analysis of $a \times b$ objects in terms of sets and arrays
Grade 2		II. Intermediate multiplicative visual conception: Contracted modeling (e.g., Fig. 8)	Repeated unit construction, where real and experientially real phenomena are interpreted in multiplicative terms	Visual to numerical fluency and vice versa, expressing real and conceptually real phenomena as multiplicative expressions	Inductive implementation of grouping actions in terms of common units	Epistemological understanding of equivalent representations for expressing the same multiplicative relationships
Grade 2	Grade 3	III. Advanced multiplicative visual conception: Number strategy-derived modeling (e.g., Figs. 6, 9)	Multiple unit construction, where a quantity expressed in multiplicative form can be expressed as the union of two or more multiplicative relationships resulting from decomposing the quantity into smaller units	Visual and numerical fluency relating multiplication as a counting task	Abductive construction of local multiplicative relationships and inductively verification on examples	Synoptical understanding of equivalent multiplicative structures
	Grade 3	IV. Place value splitting conception in a multiplicative context (e.g., Fig. 10a)	Place value processing, where a given quantity (multiplicand) is seen in a split context (i.e. as the union of several units) that needs to be repeated (according to some multiplier) and, whenever appropriate, combined, ungrouped, and regrouped	Visual to numerical fluency, with and without regrouping in both horizontal and vertical formats of multiplication	Abductive–inductive–deductive acquisition of rules and application of whole number properties for multiplying whole numbers	Synoptical understanding of the multiplication process in a split context
	Grade 3	V. Integrated multiplicative-based place value splitting conception (e.g., Fig. 10b)	Discursive representations, i.e., general statements of rules for multiplying whole numbers by a single-digit whole number	Numerical method for multiplying multi-digit whole numbers by single-digit whole numbers	Deductive action on numerical tasks that involve multiplication	Empirical support when appropriate

number of circles in each circled group. Three responses mattered, as follows: (1a) $3 + 3$; (2a) $2 + 2 + 2$; and (3a) $1 + 1 + 1 + 1 + 1 + 1$. Their teachers used the three addition expressions to introduce them to multiplication in words, as follows: (1b) 2 equal groups of 3 circles or 2 3 s; (2b) 3 equal groups of 2 circles or 3 2 s; and (3b) 6 equal groups of 1 circle or 6 1 s. The verbal responses then became the basis for conversion to the symbolic form, as follows: (1c) 2×3; (2c) 3×2; and (3c) 6×1.

Over the course of three sessions, they filled out a fourfold organizer such as the one shown in Fig. 6, which reinforced the different mathematical ways of perceiving single-digit whole-number multiplication. For Karl, the expression $a \times b$ meant that he was constructing equivalent actions that conveyed equal groups, equal rows, equal addends, and finding products. Results of the clinical interviews conducted at the end of the school year among the second-grade students indicate that 17 of them acquired Level I proficiency. For example, Mark in Fig. 7 interpreted each stipulated multiplicative expression by first identifying the common unit and then circling the indicated number of groups. When asked which multiplicative

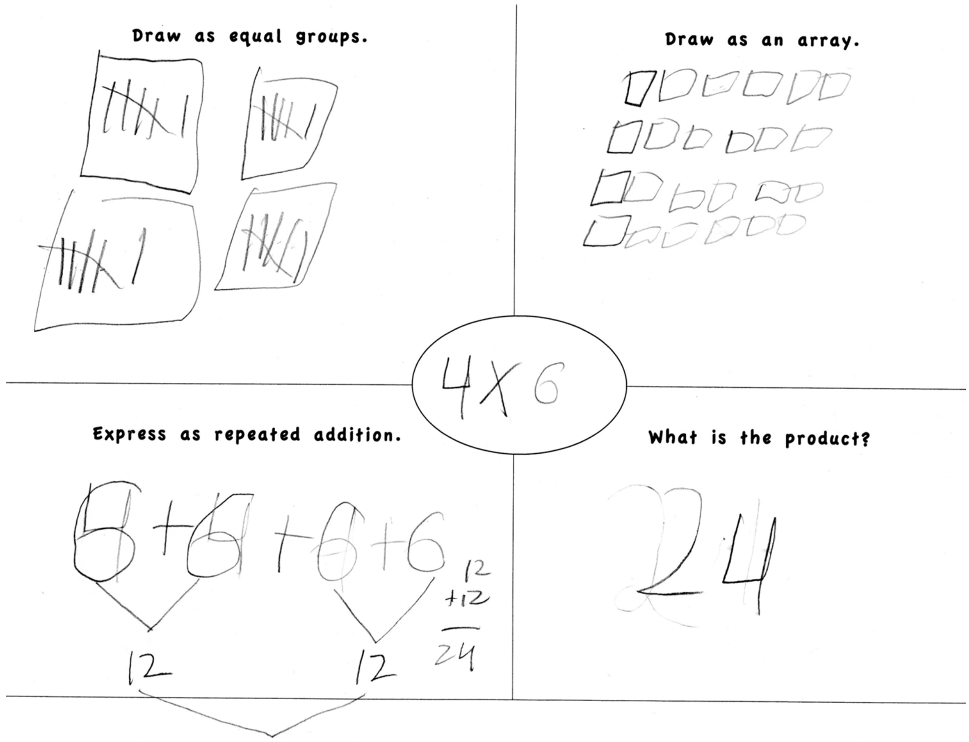

Fig. 6 Second grade Karl's multiple representational understanding of 4 × 6

expression made sense to him, Mark thought that "they [we]re all correct because they're all the same circles."

The students in second grade achieved Level II proficiency in the context of problem solving-driven activity. They inferred their converted numerical expressions as a result of initially processing and expressing the relevant problems such as the one shown in Fig. 8 in visual form. Lisa's work shows how she used her visual representation to transition to the verbal form that enabled her to obtain the answer via skip counting by 10 s.

The disposition toward Level III thinking emerged as a cognitive coping strategy that was not directly taught to the students. For example, on the second day of the teaching experiment in second grade, Karl in Fig. 6 obtained the product of 4 × 6 by taking the sum of pairs of addends, that is, 12 + 12. Karl in Fig. 9 continued to use the same decomposition strategy in the case of 6 × 7, which he interpreted as being equivalent to the sum of 14 + 14 + 14. When the students were in third grade, some of them consistently visualized sets of objects in terms of the union of two or more simpler multiplicative and/or additive expressions. Especially when a factor involved any of the digits 6, 7, 8, and 9, on their own they visually partitioned them in terms of multiplication facts that they were already familiar with. Further, the commutative and associative principles also played a significant role in Level III, which they applied in problems in which the corresponding multiplication facts needed to be broken down into simpler factors. For example, Karl's written work in Fig. 9 shows a visual-to-numerical parsing of 6 × 7 in terms of 3 × (2 × 7).

Third grader Pam's work on the multiplication compare problem in Fig. 10 illustrates Level IV thinking that enabled her to smoothly transition to Level V. Initially she employed visual processing on the place value representations as shown in Fig. 10a. The numbers under each column in Fig. 10a represented the numbers of objects that remained after a visual regrouping action on the relevant columns. Hence, when she transitioned to Level V, as shown in Fig. 10b, her multiplication algorithm merely recorded a series of processing actions, where the top numbers conveyed regrouping actions and the bottom numbers indicated the remainders after regrouping.

5.6 Conceptual processing-conversion phases for division in grades 2 and 3

Table 4 summarizes the conceptual phases for dividing whole numbers from single-digit factors to place-value splitting contexts that emerged in students' activity over 2 years. The students in second grade initially explored sharing-partitive division problems and employed two basic distributive actions that reflect Level I and II processing. Students at Level I randomly assigned objects to boxes. They associated the concept of division in terms of a correspondence between two sets, that is, from objects that needed to be distributed one by one to groups. They

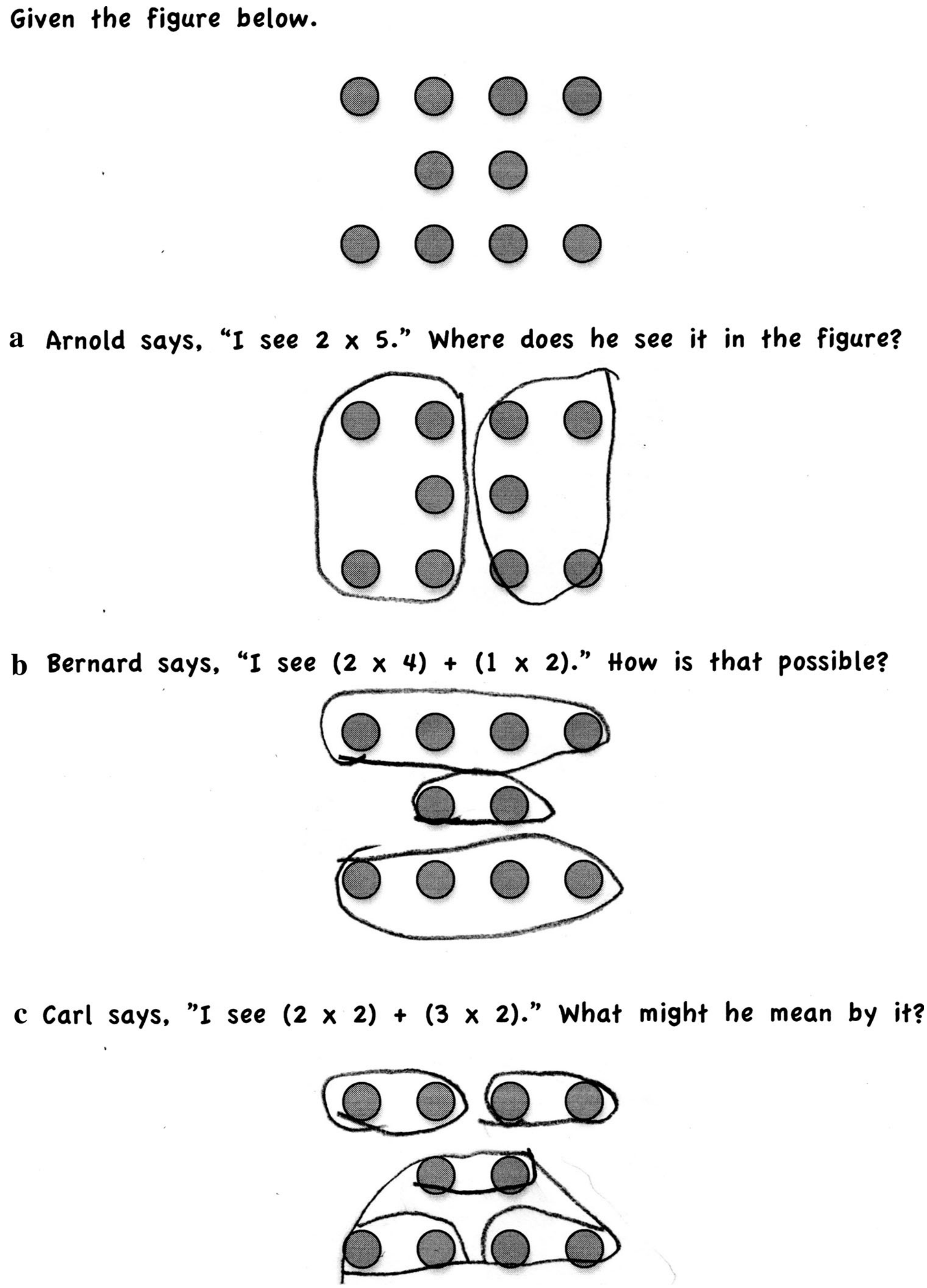

Fig. 7 Second grade Mark's work on a multiplication task administered during an end-of-the-year clinical interview

perceived quotient as referring to the number of objects in each group (i.e., internal division).

Students at Level II exhibited more systematic processing than those at Level I. Some of them initially constructed the required number of boxes horizontally or vertically and then distributed the objects one by one to each box either from left to right or top to bottom, respectively (Fig. 11a). A few others drew sticks or circles corresponding to the dividend, labeled them consecutively from 1 to n, where n corresponds to the value of the divisor, and then counted the objects that shared the same label that they inferred to be the quotient (Fig. 11b). Concrete and physical actions manifested at Levels I and II conveyed the observation that the students initially associated division with distributive action.

When the students achieved proficiency at Level II, they next tackled measurement-partitioning division problems. The most prevalent visual strategy at Level III was grouping or iterating by the divisor through either counting on, skip counting, or combinations of both strategies (Fig. 12). They stopped only when their count reached the required total. In this case, they perceived division as a systematic

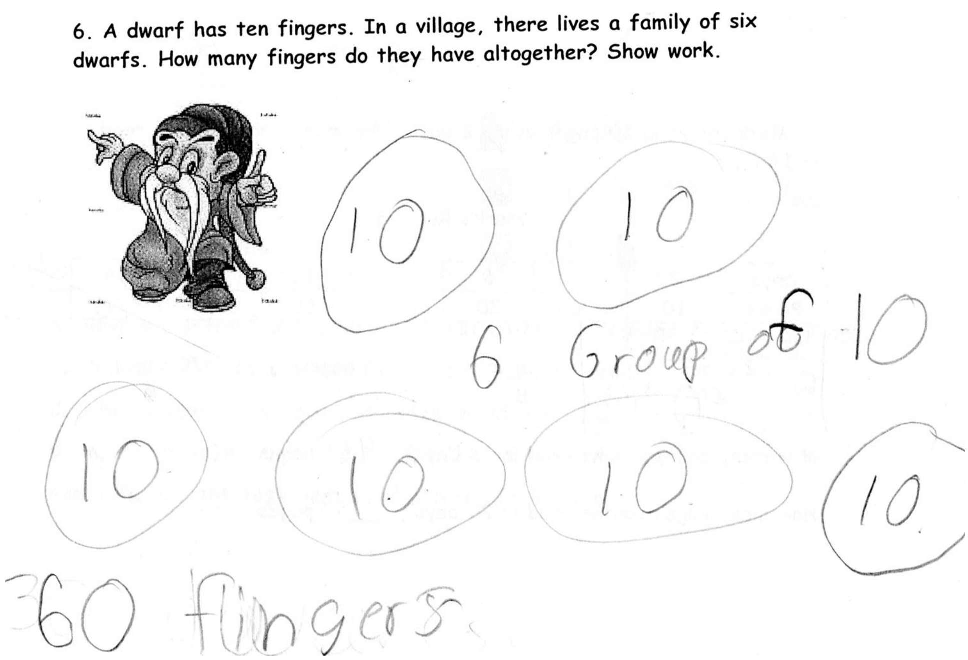

Fig. 8 Second grade Lisa's Level II analysis of a multiplication problem

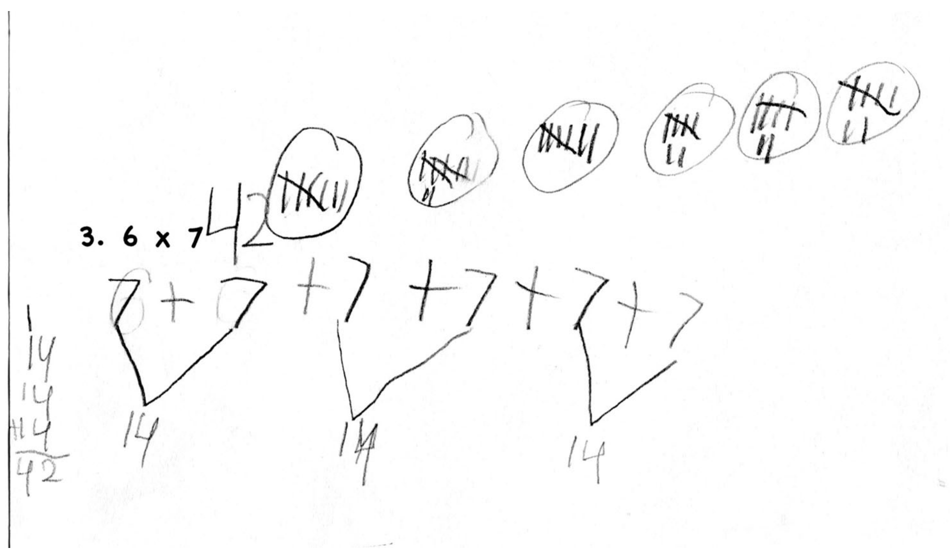

Fig. 9 Second grade Karl's Level III thinking on the expression 6 × 7

On Saturday, 1355 people visited the zoo. Three times as many people visited on Sunday than on Saturday. How many people visited the zoo on Sunday?

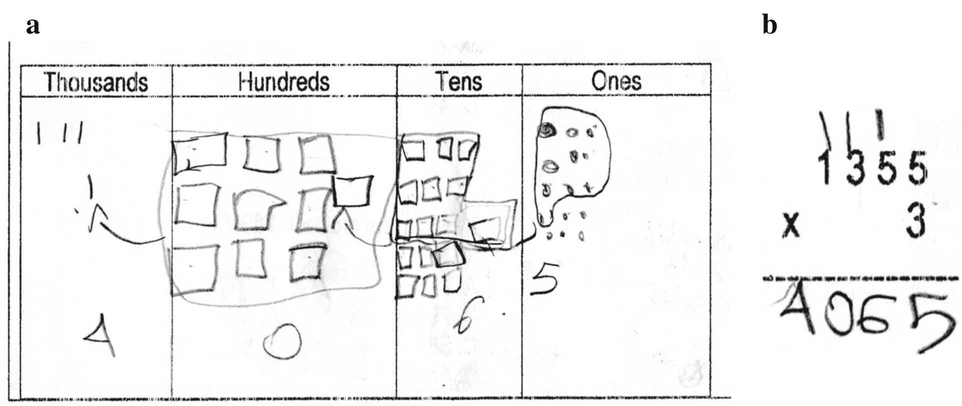

Fig. 10 Grade 3 Pam's multiplicative transformation from Level IV to Level V thinking. **a** Visual processing. **b** Numerical recording

Table 4 Conceptual visual-numerical phases for division of whole numbers in grades 2 and 3

	Grade level	Conceptual phases	Transformational processing	Transformational conversion	Relevant mathematical inference being used	Function of visual representation
Grade 2		I. Early correspondence modeling in a sharing-partitive context: internal division through distributive random action	Distributive action discernment, where a given quantity (dividend) is seen as individual units that are assigned to groups (divisor) in an unsystematic random manner	Visual fluency, informal operation (i.e., gestural and pictorial demonstrations) evident at this stage	Abductive inference of equal grouping conception through distributive action and inductive verification through examples	Epistemological understanding of the relationship between division and distributive action in a sharing-partitive context resulting in the production of a common unit
Grade 2	Grade 3	II. Foundational non-place value splitting in a sharing-partitive context: internal division through distributive systematic action (e.g., Fig. 11a, b)	Internal division construction, where a given quantity (dividend) is equally distributed over groups (divisor) and quotient refers to the common unit with remainders as a special case	Visual to numerical fluency, emergence of symbolic notation $c \div \mathrm{a} = b$, which stems from $a \times b = \mathrm{c}$	Inductive implementation of a systematic abduced distributive action yielding a common unit	Synoptical understanding of what it means to divide whole numbers into (equal) groups of common units
Grade 2	Grade 3	III. Foundational nonplace value splitting in a measurement-partitioning context: external division through grouping or iterative action (e.g., Fig. 12)	External division construction, where equal grouping action and iterating by a fixed unit (divisor) yield the required quantity (dividend) and the required quotient (i.e. number of groups or iterations) with remainders as a special case	Visual to numeric fluency relating grouping action and numerical skip counting activity in a simultaneous manner; further understanding of $c \div b = a$, which stems from $a \times b = \mathrm{c}$	Abductive inference of equivalent actions among skip counting, iterating, and grouping activity and inductively verifying the equivalence through examples	Synoptical understanding of the equivalence of equal grouping action and skip counting activity in multiplication and division in measurement-partitioning contexts
	Grade 3	IV. Formal place value splitting in division (e.g., Fig. 13a)	Place value processing, where a given quantity is seen in a separate context that needs to be internally or externally divided through repeated applications of ungrouping and regrouping of units	Visual to numerical regrouping action, where long is seen as a recording of either internal or external division processing	Deductive implementation of rules for dividing with and without regrouping in a splitting context	Synoptical understanding of the regrouping process in either internal or external division contexts
	Grade 3	V. Integrated place value splitting in division (e.g., Fig. 13b, c)	Discursive representations, i.e., general statements of rules for dividing whole numbers in any context of use and grouping action; visual processing as needed	Numerical recording of a stable process for dividing multi-digit whole numbers by single-digit divisors	Deductive action on numerical tasks that involve division	Empirical support when appropriate

correspondence between two sets, from groups of objects to the number of groups formed. In this case they viewed quotient as referring to the number of groups formed (i.e., external division).

Level IV thinking emerged in third grade when the students focused on place value division. The emerging long division algorithm at this level was once again viewed as a numerical recording of a sequence of ungrouping–

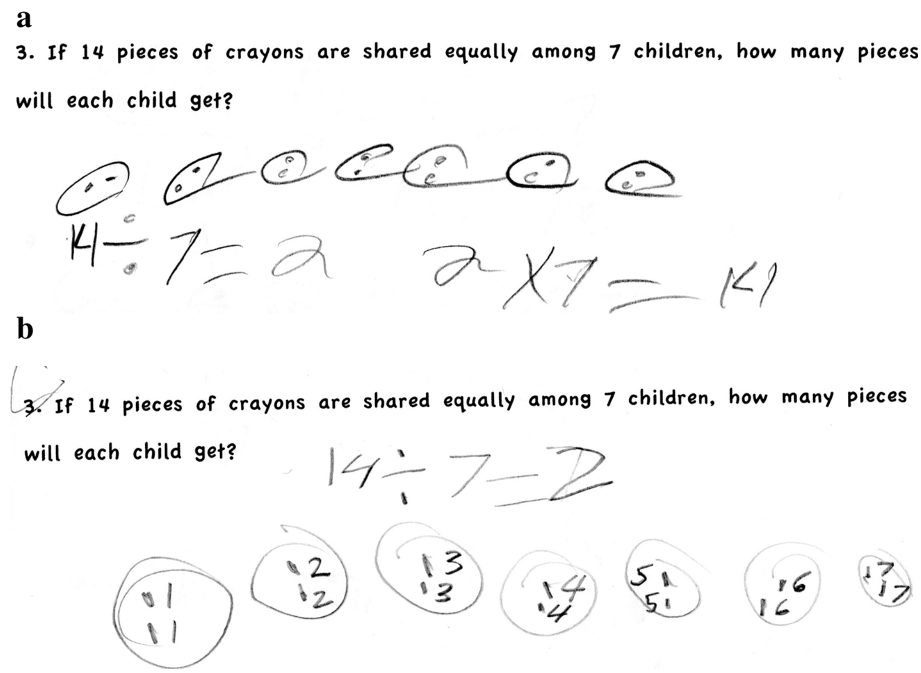

Fig. 11 Level II internal division processing in grade 2. **a** Filling a box method. **b** Same label method

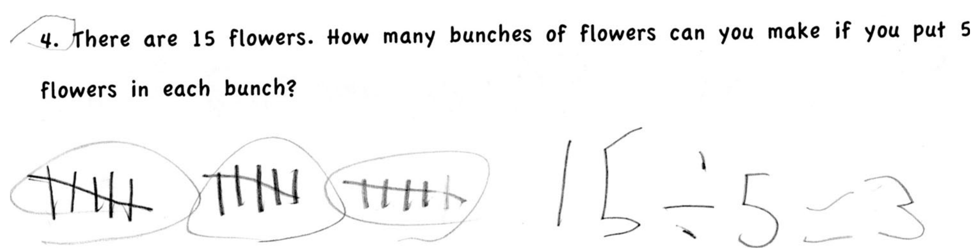

Fig. 12 Level III external division processing in grade 2

regrouping actions. For example, third-grader Mark's visual representation processing in Fig. 13a employed internal division thinking at Level II. In the case of 126 ÷ 6, when he could not divide a single (hundreds) box into six (equal) groups, he recorded it as a "0." He then ungrouped the box into 10 sticks, regrouped the sticks together, divided the sticks into 6 groups, recorded accordingly, and so on until he completed the division process for all subcollections. His numerical recording captured every step in his sequence of visual actions.

At Level V, Mark's attention shifted away from the visual form and toward the rule for division, which was accompanied by two remarkable changes in his numerical processing. In Fig. 13b, he performed division on each digit in the dividend from left to right with the superscripts indicating partial remainders that had to be ungrouped and regrouped. In Fig. 13c, he made another subtle revision that remained consistent with his earlier work and experiences. When he was asked to explain his division method, Mark claimed that "*it's like how we do adding and subtracting with regrouping, we're just doing it with division.*"

Results of the students' average proficiency percentage involving multiplying and dividing whole numbers with and without regrouping on the district and state assessments over the course of 2 years was 85 %, which was about four percentage points higher than the state average.

6 Discussion

The math drawings that the students used in mathematical activity conveyed visual non-iconic representations that they endowed with structural relationships in the process of learning about whole number operations. At the epistemological level, it did not matter to them that different figures represented the same digit in any place since they primarily attended to the structural meanings of the representations. Certainly, the visual representations had a strong initial structuralist bias in their core, which is acceptable given the fundamental nature of intentional images. At the synoptical level, the students' written solutions conveyed interesting analytical refinements as a result of progressions in their inferential and thinking abilities. Consider, for instance, the sequence of intra-representational processing phases in Gerry's visual solutions in Fig. 4a, b and Mark's numerical solutions shown in

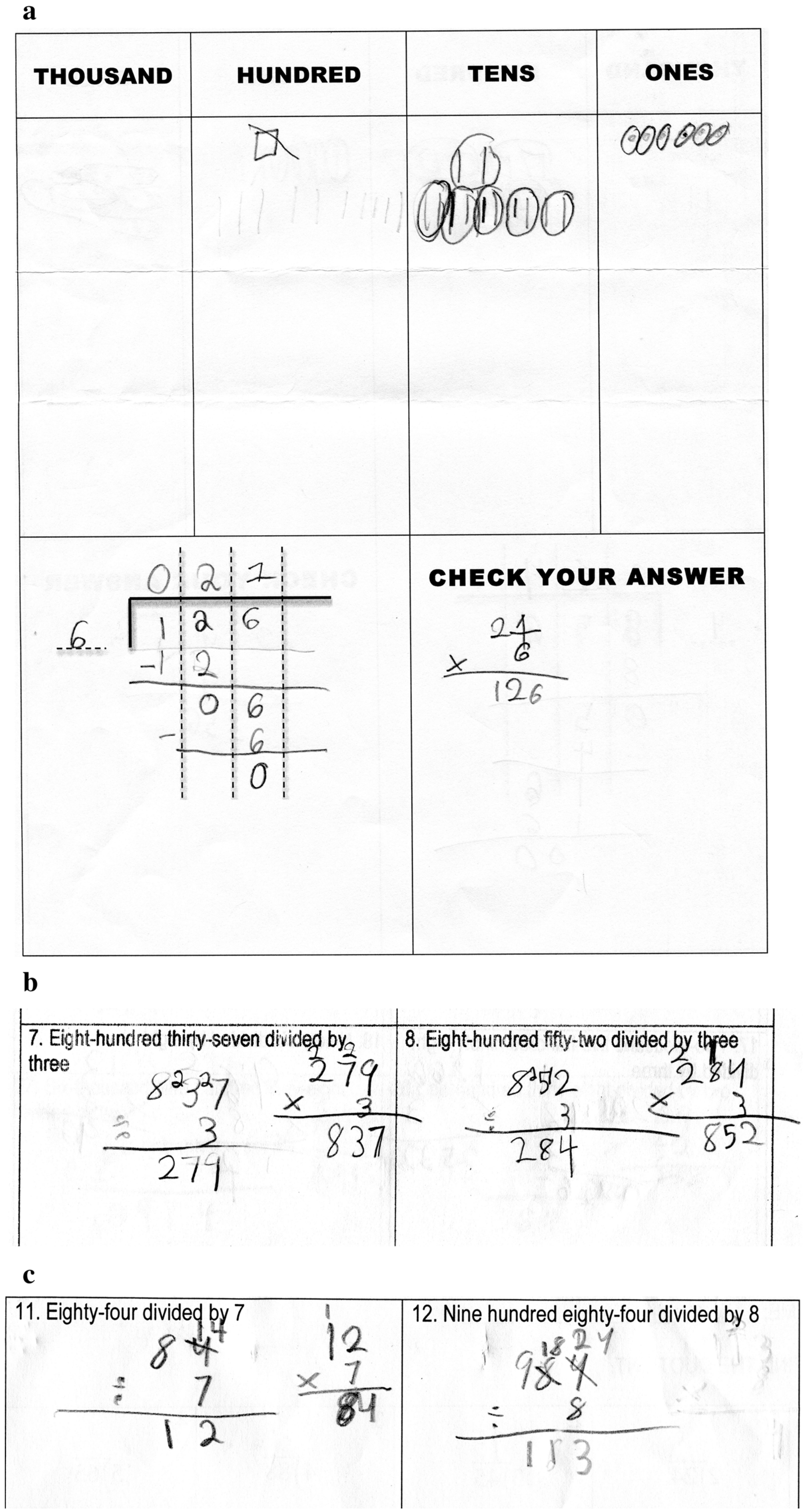

Fig. 13 Third grade Mark's division processing and conversion. **a** Mark's combined visual-numeric approach on a division problem. **b** Mark's numerical division processing that moved away from the long division format. **c** Mark's numerical division processing refinement of **b**

Fig. 13b, c and the sequence of inter-representational conversion phases in Lisa's and Pam's visual-to-numerical solutions in Figs. 3b, c and 10a, b. In these cases, changes in their inferential and thinking abilities (i.e., abductions, inductions, and deductions) enabled changes in both aspects of processing and conversion of visual and numerical representations in various conceptual phases.

Accounts of progressive conceptualization in the existing mathematics education literature at the elementary level also underscore the role of inferring and thinking in the emergence of necessary mathematical knowledge. Results of this study, however, suggest the view that students' success in the processing and conversion of their visual representations of whole number operations in numerical form would be possible only if they experience certainty that comes with deduction (Pillow et al. 2010). In this study, the visual representations initially operated within a deductive axiomatic structure, that is, math drawings as diagrams with well-defined meanings in a place-value context. When the students implemented the four arithmetical operations on them, their visual experiences enabled them to empirically infer stable and routine effects that the operations had on those diagrams. That is to say, the effects that emerged from abductive–inductive actions became the necessary grounding for generalization of rules. Further, when they experienced success in applying the rules in different arithmetical situations, it signaled the phase of deductive inferencing where they began to think about them as algorithms.

Results of this study also provide an empirical demonstration of Hiebert's theory of written symbol competence, which complements Duval's theory and implicitly draws on inferencing and thinking in various phases. The first four phases in Hiebert's theory, especially, capture quite well one way in which students' processing and conversion of whole number operations transitioned from seeing visual representations as symbols of quantities to establishing algorithms as necessary symbols for efficiency and transfer. Analyses of the students' written work and the summaries provided in Tables 2, 3, and 4 indicate the following observations.

1. *Connecting individual symbols with referents:* Phase I for all operations in Tables 2, 3, and 4 enabled the students to link both personal and math drawings with their respective signifieds. In this initial stage of notational processing, the students needed to coordinate the following three factors in order to establish meaningful and successful connection: (a) simplicity (e.g., math drawings versus faithful drawings of Dienes blocks); (b) visual-verbal-numeric transitions (e.g. Fig. 8); and (c) concrete and anticipated actions on the representations (Fig. 11). Further, the coordination took place solely at the visual level, especially in situations that involved simple manipulations resulting from the application of a certain operation. For example, the division problem tackled in Fig. 11a, b for a Phase I learner conveyed an anticipated action of distributing 14 objects into 7 boxes. Hence, in the case of internal division, the learner acquired the visual variables (Duval 2002) that mattered (i.e., objects and boxes). Another example involves Karl's multiple representations of two single-digit multiplication in Fig. 6. Across the different representations, Karl knew that his visual variables involved either constructing boxes that all had the same number of objects in each box or forming a rectangular array of equal-sized rows.

2. *Developing symbol manipulation procedures*: Phase II under addition/subtraction and multiplication in Tables 2 and 3 and phases II and III under division in Table 4 marked the beginning phases in which the students formally operated on two whole numbers. The examples they explored involved simple ungrouping contexts that enabled them to abduce and induce—and, thus, generalize—rules for combining the notations in a concrete way. For example, Gerry's work on the subtraction problem 35 − 13 in Fig. 4a had him taking away 1 stick from 3 sticks and 3 circles from 5 circles. Pam performed the same actions relative to the ungrouping tasks in Fig. 3b. Central to the conversion process in these phases for them were the parallel, faithful, and generalizable actions that enabled them to link their visual manipulations and their numerical recording of those manipulations. Generalizations of rules closed this phase in notational processing.

3. *Elaborating procedures for symbols*: In phase III under addition/subtraction and division in Tables 2 and 4 and phases III and IV under multiplication in Table 3, the students dealt with the more complicated case of regrouping. These complex situations enabled them to further test and refine the applicability and consistency of the earlier formulated rules that they established for the simple cases. Gerry's regrouping work in Fig. 4b in the case of subtraction, Pam's regrouping work on multiplication in Fig. 11a, and Mark's regrouping solution on a division task in Fig. 13a helped them extend their initial generalization rules involving ungrouping cases to regrouping situations. However, the processing and numerical recording of generalizable actions were still strongly linked to their visual representations. Generalization extensions of rules closed this phase in notational processing.

4. *Routinizing procedures for manipulating symbols*: Phase IV under addition/subtraction in Table 2 and phase V under multiplication and division in Tables 2 and 3 marked the discursive representational phase in which the students assumed the rules as hypotheses. In these phases, they applied the rules in different contexts with very minimal effort. Also, in these phases, the students have already made a full transition to the numerical format, that

is, as algorithms that they applied solely at the numerical level. The written work of Lisa in Fig. 3b, Pam in Fig. 10b, and Mark in Fig. 13b, c exemplify how they successfully employed the rules in the absence of any visual support.

7 Conclusion

In this study, notational documentation and analysis captured the emergence of elementary students' understanding of whole number operations in terms of changes within and between their visual and numerical representations. The various diagrams conveyed to them a consistent structure that provided meaningful ways to see the relationship "between empirical and deductive mathematics" (Duval 2006, p. 320). Suffice it to say, a synoptic grasp of visual representations tends to infer generalizable actions that when recorded in numerical form provide a basis for discursive representations.

Across the four arithmetical operations, the students' progressive conceptualization of whole number operations operated along similar structural phases. Further, the visual representations facilitated meaningful mathematical inference and thinking that were appropriate at their grade level. Certainly, it helped that they had an "interpretively conventional character" (Perini 2005, p. 268) that loaded them with truth-bearing and truth-preserving qualities, qualities, enabling the students to establish the relevant intentional generalizations. More importantly, however, the conversion to the numerical representations came with it the experience of deduction, which enabled them to see the power and efficiency of discursive rules for operating on numbers in a variety of situations. Of course, it helped that inter-representational conversion involved congruent relationships that consequently did not encourage them to regress to or remain at the visual level. Instead they saw value in the contracted and algorithmic form of the numerical versions and notations that also provided another way of expressing generalizations and discursively-drawn representations.

One issue that is worth pursuing in a follow-up study involves understanding how children reason about, and become fully aware of, the certainty of their inferences regarding whole number operations from the processing to the conversion phase, where reasoning pertains to explaining an inference in the context of some appropriate norm (Moshman 2004, pp. 223–224). Since this study primarily drew on notational development and understanding, it also implied to some extent how children tend to "assimilate" information to some "inference schema, but do not explicitly think about the process of drawing conclusions from the premises" (Pillow et al. 2010, p. 214).

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