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Changing the Face of Arithmetic: Teaching Children Algebra

Research studies of classes in which the arithmetic curriculum has taken an algebraic approach provide stunning evidence that elementary students can—

- intuitively understand basic algebraic properties relevant to solving equations such as the addition property of equality (i.e., identical units added to both sides of the equation do not affect the resulting equality);
- develop situated and consistent forms of algebraic notation and rules;
- solve simple equations using a variety of empirical strategies such as trial and error;
- generalize simple linear patterns from a table of values;
- make sense of the graph of a linear function; and
- develop an intuitive notion of functions as rules of correspondence involving objects or elements in a sequence. (Dougherty 2005; Schliemann et al. 2003)

The basic question that must be addressed at this time is *why* is there an interest in integrating algebra in the elementary arithmetic curriculum (*Algebraic Thinking* 1997; NCTM 2000) and, more important, what benefits await children who develop algebraic reasoning as early as first grade? Based on findings from research, significant differences between arithmetic and algebra might explain why older children tend to experience difficulty in learning algebraic ideas. Investigations done with older children show they have difficulty transitioning to algebra from an arithmetic-only curriculum because arithmetic deals mostly with particular numbers, quantities, and operations acquired by rote, whereas algebra focuses on variables, functions, and invariant relationships and structures (Brizuela and Schliemann 2003). Another possible source of difficulty is the perception that algebra requires a certain level of abstraction and mental maturity that an arithmetic curriculum does not sufficiently address (Herscovics and Linchevski 1994). Also, a mistaken assumption exists that if children were to develop an understanding of arithmetical operations, they would induce the corresponding arithmetical structures that are necessary for and preparatory to algebraic thinking (Warren 2004).

Suggestions for Teaching an Algebrafied Arithmetic Curriculum

Mathematics education researchers who advocate an early algebra curriculum are telling us that it is feasible to "algebrafy" arithmetic by loading arithmetical tasks with algebraic meaning that is appropriate to young children. Based on current research findings, the following recommendations target different aspects of elementary algebraic thinking.

Teach number systems in such a way that students are aware that inherent properties or relations exist that must be articulated mathematically. When we teach children the arithmetic of whole numbers, we may focus instruction on possible mathematical properties or relationships that they can draw from individual objects. They develop the view early in their mathematical experiences that doing mathematics involves searching for invariant properties or relationships that are ultimately independent of the objects in which they have been drawn initially.

For example, play a number game in which children guess a rule for the relationship between pairs of numbers in the following given set of paired input-output values: (3, 6), (7, 10), (5, 8), ... The given sequence can be expressed in different ways,

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Edited by **Alfinio Flores,** *alfinio@asu.edu, Arizona State University Department of Curriculum and Instruction, Tempe, AZ 85287-0911. This department describes research and demonstrates its importance to practicing classroom teachers. Readers are encouraged to send manuscripts appropriate for this section by accessing* **tcm.msubmit.net**.

such as by mapping or using a table. Carraher, Schliemann, and Brizuela (2000) report that eighteen third graders in their study were successful in obtaining the correct relationship (for each ordered pair the second number is three more than the first), and that they understood the rule to work with all numbers, not just the given ones. The teacher in this experiment provided the initial pairs and led the class to a rule by asking a series of questions, such as the following:

- "If I start from 5, I'm going to go to what?"
- "If I start from *n*, then I have to go to what?"
- "So how am I going to write that down?"

Asking children to formulate a rule will help them make sense of which properties or relationships stay the same or change, and also what variables are and the role they play in explicitly expressing and connecting relationships among numbers and quantities. Thus a shift in instructional activity takes place, from merely performing operations on numbers to establishing numerical properties and patterns of relationships among numbers.

Teach children to value informal and formal representations. One goal of instruction is to bridge children's own symbols with the formal representational systems valued by the wider mathematical community. Notations, symbols, and all other forms of representation organize children's thinking and understanding. Informal representations are "not lesser means of doing mathematics, but the very material basis of sense-making" (Meira 2002, p. 102). Representations are a form of written manifestation of what and how children are thinking, and they help children decide what and how to think. For instance, when children see numbers from problems, many of them are predisposed to think or reason in computational terms without considering what analysis must be done first, including possible relationships that they need to establish prior to making any calculation. In particular, some elementary children associate the equals sign with "doing something" (Saenz-Ludlow and Walgamuth 1998), that is, as calculating numbers on one side and stating the answer on the other side. It is not seen in the context of the relationship "is the same as" (Falkner, Levi, and Carpenter 1999, p. 232). Hence we bear the responsibility of providing children with situations that allow them to expand and enrich their understanding of symbols and notations and to transition to more formal representations. A basic goal of elementary mathematical

instruction is to help students see what representations are, what representations are possible, why they exist, why so many exist, and which ones will make the most mathematical sense. Warren's (2004) work with 8-year-old children in five elementary schools in Australia illustrates how representations tend to influence students' abilities to generalize that result in different types and levels of generalization. It seems there is a relationship between representational competence and facility in making generalizations, an important skill in algebra.

An activity that can engender talk among children about representations is the problem below, drawn from Blanton and Kaput's (2004) work with prekindergarten to fifth-grade students.

> Suppose you were at a dog shelter and you wanted to count all the dog eyes you saw. If there was one dog, how many eyes would there be? What if there were two dogs? Three dogs? 100 dogs? Do you see a relationship between the number of dogs and the total number of

eyes? How would you describe this relationship? How do you know this works? (Blanton and Kaput 2004, p. 136)

Blanton and Kaput report that the participating children provided different representations for the problem. Furthermore, the solutions they generated reflected different perspectives of mathematical thinking, from additive (counting the number of eyes involves counting by twos) to multiplicative (double the number of dogs).

Teach functions so that children can begin to develop a predisposition for algebraic modeling. Elementary students are capable of functional thinking, and the rules they describe contain additive and multiplicative relationships. Exposing them to functions in their early mathematical experiences provides them with an opportunity to set up rules of correspondence between two objects or elements in a set. We may initially use tables that consist of input and output values to help children organize their work. Later they can choose to either use the table/chart method or develop other forms of functional representation. When we teach the four fundamental operations from a functional standpoint, a shift in students' thinking takes place from calculating results to figuring out rules relevant to the four fundamental operations. Thus they begin to explore how operations can be perceived as not merely "a process that produced a product, the answer" but as a "process of change" (Warren 2004, p. 423). We should also encourage them to develop an intuitive, visual understanding of functions and to see how useful algebraic notations are in expressing relationships among whole numbers.

For example, in teaching the multiplication table as a function, students can be introduced to mathematical notions such as direct variation, domain and range, coordinate points in a coordinate plane, and linear functions. When the focus of learning is on functions, elementary students can then begin to view the following tasks as natural extensions of their work with numbers: obtaining a rule and a formula for a table of values or sequence of numbers; plotting points in a Cartesian system; and observing how sets of coordinates appear as points lying on a straight line. Carraher, Schliemann, and Brizuela (2000) report that some third graders in their study were successful in understanding the form $y = 2x + 1$ given a table of values in which the x column contains numbers from 1 to 10, 20, 30, 100, as well as n. The y column contains the numbers 3, 5, 7, and 9. As the students were computing the output values for, say, $3n + 2$, at least one of them interpreted the expression to mean "It's like doing the 3's table and adding 2 more" (p. 15). In fact, toward the end of the teaching experiment, the students in their study were successful in drawing closed forms for certain sequences of whole numbers. Furthermore, they interpreted arithmetical operations far beyond the usual notion that operations were merely procedures for combining individual numbers.

Elementary students can do more than understand an explicit rule. They can also infer a functional relationship and derive a rule from paired sets of input-output values. For example, in the activity "Guess My Rule," Carraher and Earnest (2003) report that eighteen third-grade students were successful in coming up with linear patterns from a given set of paired input-output values. Initially, the students worked in small groups to formulate their own linear rules and to compute particular cases so that others would be able to guess a rule on the basis of the input-output values. When a group offered one million as an input and twenty-one million and one as the output, a student named Cristian was able to state a correct rule, that is, "n times 20 plus 1." The rules that the students developed varied in difficulty. One group suggested the rule "k times 2 minus 2," which did not seem to cause too much trouble for the guessers. Carraher and Earnest's research also surfaces an interesting point with regard to the difficulty elementary students may have in making sense of equivalent expressions. For example, a considerable amount of classroom discussion arose when Cristian's guess did not appear to be the same as the group's formula that took the form "$n \times 5 \times 4 + 1$." This grappling with equivalence arose once more when the students had to deal with whether $k \times 2$ and $k + k$ meant the same thing, which the class never fully resolved.

Teach arithmetic problems and create learning situations that require elementary students to think about mathematical relationships first before any computation is done. All too often, mathematical instruction and textbook problems in arithmetic acculturate children to particular ways of acting that "condition" them to employ computational procedures, even if that is not the first thing that they need to do. The task in **figure 1** is a good example of an open-ended problem that focuses on relationships among quantities. It deals with asking students to first compare heights between and among Tom, Maria, and Leslie and then to compute values in order to verify their arguments. Also, ask students to analyze relational statements such as "Ngoc has twice as much money as Pedro," using tables and graphs, and gradually add a condition or two that will allow them to generate different approaches to solving the problems (Schliemann et al. 2003).

Davydov (1975) suggests that instead of starting with numbers, it might be better for young children to first explore basic conceptual ideas such as sets, equivalences, and powers. For instance, ask them to compare attributes of everyday objects based on some measurable property such as length or area. Encourage them to describe the comparison by way of diagrams and relational sentences, using variables to indicate that letters stand for quantities being compared and not the objects themselves. Capitalize on the use of concrete objects and discuss the significance of various modes of gestures and actions that they perform with the objects. This "prenumeric" stage of learning will provide young children with meaningful mathematical opportunities of, say, manipulating variable expressions, equations, or inequalities using appropriate properties or employing variables for naming and labeling particular relationships. In the interview transcript below, Mia, a first grader in Dougherty's (2005) study, explains to the interviewer the importance of using variables as a way of expressing and communicating to others an explicit relationship between the volumes of two different bottles.

> You can't just say the volume in the red bottle is more than the volume in another bottle. But we could name the volumes like C and volume E. Then it doesn't matter what bottle it's in, it's the quantity that counts. [Then Mia wrote the following expression: C > E.] (Dougherty 2005, p. 5)

Research done with first graders by Dougherty (2005) and her colleagues shows that the children in their studies were successful in obtaining such generalizations, and they used the generalizations to further investigate particular instances.

Teach arithmetic problems that encourage multiple solutions. This suggestion is novel, particularly among those of us who perceive arithmetical tasks as falling within a "single type, one answer only" problem category. We all need to be aware that valid multiple solutions to a problem are an indication that different, but equally correct, interpretations of, or approaches to, solving the problem are possible. The heights problem in **figure 1** is typical of open-ended problems that we may consider authentic in the sense that it invites multiple solutions and multiple answers. When eighteen third graders were asked to solve the given task, Carraher, Schliemann, and Brizuela (2000) report that twelve children assigned specific values for each child's height that enabled them to construct a pictorial relationship, and the remaining six believed that generating different solutions was possible. The accompanying questions in **figure 2** are aimed at encouraging children to test different possibilities instead of merely seeking out only one solution or answer.

Figure 1

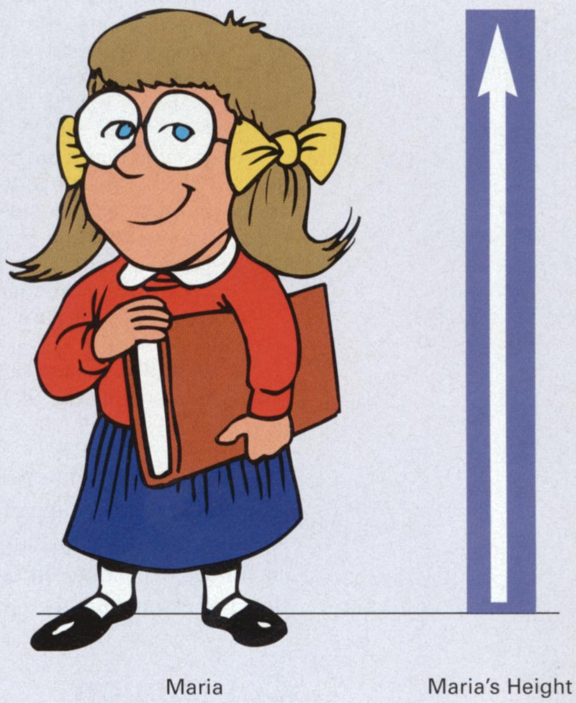
Comparing the heights of Tom, Maria, and Leslie (Carraher, Schliemann, and Brizuela 2000)

Tom is 4 inches taller than Maria.

Maria is 6 inches shorter than Leslie.

Draw Tom's height, Maria's height, and Leslie's height.

Show what the numbers 4 and 6 refer to.



Exposing children early in their mathematical experiences to open-ended problems predisposes them to the view that problems can be represented in several different ways. We emphasize once more how different solutions and representations of children are related to the manner in which they use and understand notations and symbols. Some elementary students tend to perceive variables and unknowns as the same concept. This overgeneralization arises from their limited experiences with problems in arithmetic (such as "Solve for n in $13 + 7 = n$") that use a placeholder for a particular, fixed value that must be computed in some way. However, other more complex, open problems (such as what

Figure 2

Possible questions to accompany the task in figure 1

1. Why did you use that particular drawing?
2. What can you tell me about the heights of Maria, Tom, and Leslie from your drawing? How did you figure out the heights?
3. When you thought about making a drawing of the heights of Tom, Maria, and Leslie, did you need to know what the height of each person was? Why? Why not?
4. What do the numbers 4 and 6 refer to in your drawing?
5. Can you figure out their heights in a different way? How would you do it?
6. Will you be able to tell for sure who is the tallest among Maria, Tom, and Leslie? What about the shortest? How could you figure it out?
7. Is there only one answer to this problem? Why? Why not?
8. If there is more than one answer, can you show me how you figure out a different answer?
9. If there are more than two answers, can you figure out how many answers there are in all?
10. If you know Leslie's height, will you be able to figure out the heights of Tom and Maria? Why? Why not?
11. What if you start out knowing what Tom's height is—will you be able to figure out the heights of Maria and Leslie? If yes, how? If no, why not?

values can be substituted for a and b in $a = b + 2$) require an understanding of a letter as a symbol representing a variable quantity that can take on different values depending on the contexts or conditions stated or assumed in the problems.

Conclusion

Research on elementary children's mathematical thinking provides strong evidence that such learners are indeed capable of reasoning far beyond what we and, in general, our societies "normally" assume they can and cannot do. Introducing algebra in the elementary school mathematics curriculum does not mean doing away with traditional, foundational concepts, processes, and operations that all children must have in order to be arithmetically proficient. What early algebra seeks to accomplish is to take a second look at arithmetical topics "in a new light and with a new set of attitudes" (Carraher, Schliemann, and Brizuela 2000, p. 21). The proposal to "algebrafy" arithmetic captures the essence of this purpose. That is to say, an algebrafied arithmetic encourages children to think in terms of multiple relationships within the context of real or experientially real problem-solving situations. Furthermore, such algebrafication involves asking our elementary students to think about mathematical objects such as whole numbers, *not as mere objects per se* with known procedures for combining them but as objects whose mathematical structures can be determined rather easily. In an important sense, an algebrafied arithmetic reorients students' mathematical thinking toward relationships, including the attainment of powerful algebraic skills such as patterning and generalizing (Moses 1997). Algebrafying arithmetic is not an attempt to teach young children high school algebra. Learning arithmetic should be as much about acquiring procedures as it is about developing an understanding of the underlying general principles (Carpenter, Franke, and Levi 2003).

Teaching early algebra will require time, effort, and a different perspective on the way we ask children to do arithmetic. At the very least, their repertoire of arithmetical skills must include facility in both numerical and generalized reasoning. This means the nature of what and how we ask must be changed. For example, instead of asking students to find "the" answer to a problem such as $5 \times 7 = n$, we may want to pose questions such as "What number can I replace n by and make this a true statement?" to treat the statement in an algebraic way (Usiskin 1999, p. 6). Also, we may develop tasks such as constructing true/false and open number sentences in which students are allowed to argue and to use variables to represent their ideas (Carpenter, Franke, and Levi 2003).

Kieran and Chalouh (1993) note that efforts at assisting young students to transition from arithmetic to algebra must take into account how students can be provided with meaningful grade-level scaffolds in "using letters to represent numbers" and in being "explicitly aware of the mathematical method that is being symbolized by the use of both numbers and letters" (p. 179). In the case of method, children must understand what operations do to individual numbers, how the operations make answers possible, and what notations and algebraic representations are used to describe such operations. Letters pertain to either a specific value (for example, those that pertain to solving linear equations) or a range of values (for example, those that are used to express generalization formulas). Kieran and Chalouh (1993) report that children seem to have more trouble making sense of letters as representing a range of values rather than a single value. They suggest that children's algebraic experiences should involve exposure to problems that target both uses of letters. For example, in teaching children to calculate a specific value, say, in the context of solving linear equations, using a "covering up" strategy and then following it with instruction using formal methods may be more effective for students than teaching only the latter. Such a strategy involves asking teachers to initially verbalize and for students to make numerical sense of questions such as "What number plus 5 gives 15?" in the case of the equation * + 5 = 15 before the students are taught the formal strategies. Insofar as teaching children to understand the significance of letters as representing a range of numbers, they should be exposed to problems that encourage them to see how their notion of variables can be extended so that they are employed as a vehicle for expressing mathematical truths in a general way but still within an arithmetical context.

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